RATIONAL HOMOTOPY THEORY

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INTRODUCTION

Homotopy theory is the study of topological spaces with homotopy equivalences. Recall that a homeomorphism is given by two maps $f : X \rightleftarrows Y : g$ such that the both compositions are equal to identities. A homotopy equivalence weakens this by requiring that the compositions are only homotopic to the identities. Equivalent spaces will often have equal invariants.

Typical examples of such homotopy invariants are the homology groups $H_n(X)$ and the homotopy groups $\pi_n(X)$. The latter is defined as the set of continuous maps $S^n \to X$ up to homotopy. Despite the easy definition, the groups $\pi_n(S^k)$ are very hard to calculate and much of it is even unknown as of today.

In rational homotopy theory one simplifies these invariants. Instead of considering $H_n(X)$ and $\pi_n(X)$, we consider the rational homology groups $H_n(X; \mathbb{Q})$ and the rational homotopy groups $\pi_n(X) \otimes \mathbb{Q}$. In fact, these groups are $\mathbb{Q}$-vector spaces, and hence contain no torsion information. This disadvantage of losing some information is compensated by the fact that it is easier to calculate these invariants.

The first steps towards this theory were taken by Serre in the 1950s. In [Ser53] he successfully calculated the torsion-free part of $\pi_n(S^k)$ for all n and k. The outcome was remarkably easy and structured.

The fact that the rational homotopy groups of the spheres are so simple led other mathematician believe that there could be a simple description for all of rational homotopy theory. The first to successfully give an algebraic model for rational homotopy theory was Quillen in the 1960s [Qui69]. His approach, however, is quite complicated. The equivalence he proves passes through four different model categories. Not much later Sullivan gave an approach which resembles some ideas from de Rahm cohomology [SR05], which is of a more geometric nature. The theory of Sullivan is the main subject of this thesis.

The most influential paper is from Bousfield and Gugenheim which combines Quillen's abstract machinery of model categories with the approach of Sullivan [BG76]. Being only a paper, it does not contain a lot of details, which might scare the reader at first.

There is a much newer book by Félix, Halperin and Thomas [FHT01]. This book covers much more than the paper from Bousfield and Gugenheim but does not use the theory of model categories. On one hand, this makes the proofs more elementary, on the other hand it may obscure some abstract constructions. This thesis will provide a middle ground. We will use model categories, but still provide a lot of detail.

After some preliminaries, this thesis will start with some of the work from Serre in Chapter 2. We will avoid the use of spectral sequences. The theorems are more specific than we actually need and there are easier, more abstract ways to prove what we need. But these theorems in their current form are nice on their own rights, and so they are included in this thesis.

The next chapter (Chapter 3) describes a way to localize a space directly, in the same way we can localize an abelian group. This technique allows us to consider ordinary homotopy equivalences between the localized spaces, instead of rational equivalences, which are harder to grasp.

The longest chapter is Chapter 4. In this chapter we will describe commutative differential graded algebras and their homotopy theory. One can think of these objects as rings which are at the same time cochain complexes. Not only will we describe a model structure on this category, we will also explicitly describe homotopy relations and homotopy groups.

In Chapter 5 we define an adjunction between simplicial sets and commutative differential graded algebras. It is here that we see a result similar to the de Rahm complex of a manifold.

Chapter 6 brings us back to the study of commutative differential graded algebras. In this chapter we study to so called minimal models. These models enjoy the property that homotopically equivalent minimal models are actually isomorphic. Furthermore their homotopy groups are easily calculated.

The main theorem is proven in Chapter 7. The adjunction from Chapter 5 turns out to induce an equivalence on (subcategories of) the homotopy categories. This unifies rational homotopy theory of spaces with the homotopy theory of commutative differential graded algebras.

Finally we will see some explicit calculations in Chapter 8. These calculations are remarkable easy. To prove for instance Serre's result on the rational homotopy groups of spheres, we construct a minimal model and read off their homotopy groups. We will also discuss related topics in Chapter 9 which will conclude this thesis.

PRELIMINARIES AND NOTATION We assume the reader is familiar with category theory, basics from algebraic topology and the basics of simplicial sets. Some knowledge about differential graded algebra (or homological algebra) and model categories is also assumed, but the reader may review some facts on homological algebra in Appendix A and facts on model categories in Appendix B.

We will fix the following notations and categories.

- $\mathbb{k}$ will denote a field of characteristic zero. Modules, tensor products, ... are understood as $\mathbb{k}$-vector spaces, tensor products over $\mathbb{k}$, ...

- $\mathbf{Hom}_{\mathbf{C}}(A, B)$ will denote the set of maps from A to B in the category $\mathbf{C}$. The subscript $\mathbf{C}$ may occasionally be left out.

- **Top**: category of topological spaces and continuous maps. We denote the full subcategory of r-connected spaces by $\mathbf{Top}_r$, this convention is also used for other categories.

- **Ab**: category of abelian groups and group homomorphisms.

- **sSet**: category of simplicial sets and simplicial maps. More generally we have the category of simplicial objects, **sC**, for any category **C**. We have the homotopy equivalence $|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S$ to switch between topological spaces and simplicial sets.

- $\mathbf{DGA}_{\mathbb{k}}$: category of non-negatively differential graded algebras over $\mathbb{k}$ (as defined in the appendix) and graded algebra maps. As a shorthand we will refer to such an object as *dga*. Furthermore $\mathbf{CDGA}_{\mathbb{k}}$ is the full subcategory of $\mathbf{DGA}_{\mathbb{k}}$ of commutative dga's (*cdga*'s).

CONTENTS

Part I

BASICS OF RATIONAL HOMOTOPY THEORY

1 RATIONAL HOMOTOPY THEORY

In this section we will state the aim of rational homotopy theory. Moreover we will recall classical theorems from algebraic topology and deduce rational versions of them.

In the following definition *space* is to be understood as a topological space or a simplicial set.

Definition 1.0.1. A 0-connected space X with abelian fundamental group is a *rational space* if

$$\pi_i(X) \text{ is a } \mathbb{Q}\text{-vector space} \quad \forall i > 0.$$

The full subcategory of rational spaces is denoted by $\mathbf{Top}_{\mathbb{Q}}$ (or $\mathbf{sSet}_{\mathbb{Q}}$ when working with simplicial sets).

Definition 1.0.2. We define the *rational homotopy groups* of a 0-connected space X with abelian fundamental group as:

$$\pi_i(X) \otimes \mathbb{Q} \quad \forall i > 0.$$

In order to define the tensor product $\pi_1(X) \otimes \mathbb{Q}$ we need that the fundamental group is abelian, the higher homotopy groups are always abelian. There is a more general approach using *nilpotent groups*, which admit $\mathbb{Q}$-completions [BG76]. Since this is rather technical we will often restrict ourselves to spaces as above or even simply connected spaces.

Note that for a rational space X, the ordinary homotopy groups are isomorphic to the rational homotopy groups, i.e. $\pi_i(X) \otimes \mathbb{Q} \cong \pi_i(X)$.

Definition 1.0.3. A map $f : X \to Y$ is a *rational homotopy equivalence* if $\pi_i(f) \otimes \mathbb{Q}$ is a linear isomorphism for all $i > 0$.

Definition 1.0.4. A map $f : X \to X_0$ is a *rationalization* if X_0 is rational and f is a rational homotopy equivalence.

Note that a weak equivalence is always a rational equivalence. Furthermore if $f : X \to Y$ is a map between rational spaces, then f is a rational homotopy equivalence if and only if f is a weak equivalence.

The theory of rational homotopy is the study of spaces with rational equivalences. Quillen defines a model structure on simply connected simplicial sets with rational equivalences as weak equivalences [Qui69]. This means that there is a homotopy category $\mathbf{Ho}^{\mathbb{Q}}(\mathbf{sSet}_1)$. However we will later prove that every simply connected space has a rationalization, so that $\mathbf{Ho}_{\mathbb{Q}}(\mathbf{sSet}_1) = \mathbf{Ho}(\mathbf{sSet}_{1,\mathbb{Q}})$ are equivalent categories. This means that we do not need the model structure defined by Quillen, but we can just restrict ourselves to rational spaces with ordinary weak equivalences.

1.1 CLASSICAL RESULTS FROM ALGEBRAIC TOPOLOGY

We will now recall known results from algebraic topology, without proof. One can find many of these results in basic text books, such as [May99, Dol72].

Theorem 1.1.1. *(Relative Hurewicz Theorem) For any inclusion of spaces $Y \subset X$ and all $i > 0$, there is a natural map*

$$h_i : \pi_i(X, Y) \to H_i(X, Y).$$

If furthermore (X, Y) is n-connected ($n > 0$), then the map h_i is an isomorphism for all $i \leq n + 1$.

Theorem 1.1.2. *(Long Exact Sequence of Homotopy Groups) Let $f : X \to Y$ be a Serre fibration, then there is a long exact sequence:*

$$\cdots \xrightarrow{\partial} \pi_i(F) \xrightarrow{i_*} \pi_i(X) \xrightarrow{f_*} \pi_i(Y) \xrightarrow{\partial} \cdots \to \pi_0(Y) \to *,$$

where F is the fiber of f.

Using an inductive argument and the previous two theorems, one can show the following theorem (as for example shown in [GM13]).

Theorem 1.1.3. *(Whitehead Theorem) For any map $f : X \to Y$ between 1-connected spaces, $\pi_i(f)$ is an isomorphism $\forall 0 < i < r$ if and only if $H_i(f)$ is an isomorphism $\forall 0 < i < r$. In particular we see that f is a weak equivalence if and only if it induces an isomorphism on homology.*

The following two theorems can be found in textbooks about homological algebra such as [Wei95, Rot09]. Note that when the degrees are left out, $H(X; A)$ denotes the graded homology module with coefficients in A.

Theorem 1.1.4. *(Universal Coefficient Theorem) For any space X and abelian group A, there are natural short exact sequences*

$$0 \to H_n(X) \otimes A \to H_n(X; A) \to \mathrm{Tor}(H_{n-1}(X), A) \to 0,$$

$$0 \to \mathrm{Ext}(H_{n-1}(X), A) \to H^n(X; A) \to \mathbf{Hom}(H_n(X), A) \to 0.$$

Theorem 1.1.5. *(Künneth Theorem) For spaces X and Y, there is a short exact sequence*

$$0 \to H(X;A) \otimes H(Y;A) \to H(X \times Y;A) \to \mathrm{Tor}_{*-1}(H(X;A), H(Y;A)) \to 0\,,$$

where the $H(X;A)$, $H(X;A)$ and their tensor product are considered as graded modules. The Tor group is graded as $\mathrm{Tor}_n(A,B) = \bigoplus_{i+j=n}(A_i, B_j)$.

1.2 CONSEQUENCES FOR RATIONAL HOMOTOPY THEORY

The latter two theorems have a direct consequence for rational homotopy theory. By taking $A = \mathbb{Q}$ we see that the torsion groups vanish. We have the immediate corollary.

Corollary 1.2.1. *We have the following natural isomorphisms in rational homology, and we can relate rational cohomology naturally to rational homology*

$$\begin{aligned} H_*(X) \otimes \mathbb{Q} &\xrightarrow{\cong} H_*(X;\mathbb{Q}), \\ H_*(X;\mathbb{Q}) \otimes H_*(Y;\mathbb{Q}) &\xrightarrow{\cong} H_*(X \times Y;\mathbb{Q}), \\ H^*(X;\mathbb{Q}) &\xrightarrow{\cong} \mathbf{Hom}(H_*(X);\mathbb{Q}). \end{aligned}$$

The long exact sequence for a Serre fibration also has a direct consequence for rational homotopy theory.

Corollary 1.2.2. *Let $f : X \to Y$ be a Serre fibration with fiber F, all 0-connected with abelian fundamental group, then there is a natural long exact sequence of rational homotopy groups:*

$$\cdots \xrightarrow{\partial} \pi_i(F) \otimes \mathbb{Q} \xrightarrow{i_*} \pi_i(X) \otimes \mathbb{Q} \xrightarrow{f_*} \pi_i(Y) \otimes \mathbb{Q} \xrightarrow{\partial} \cdots\,.$$

In the next sections we will prove the rational Hurewicz and rational Whitehead theorems. These theorems are due to Serre [Ser53].

2 SERRE THEOREMS MOD $\mathcal{C}$

In this section we will prove the Whitehead and Hurewicz theorems in a rational context. Serre proved these results in [Ser53]. In his paper he considered homology groups 'modulo a class of abelian groups'. In our case of rational homotopy theory, this class will be the class of torsion groups.

Definition 2.0.3. A class $\mathcal{C} \subset \mathbf{Ab}$ is a *Serre class* if

- for all exact sequences $0 \to A \to B \to C \to 0$ if two abelian groups are in $\mathcal{C}$, then so is the third,
- for all $A \in \mathcal{C}$ the tensor product $A \otimes B$ is in $\mathcal{C}$ for any abelian group B,
- for all $A \in \mathcal{C}$ the Tor group $\mathrm{Tor}(A, B)$ is in $\mathcal{C}$ for any abelian group B, and
- for all $A \in \mathcal{C}$ the group homology $H_i(A; \mathbb{Z})$ is in $\mathcal{C}$ for all positive i.

Serre gave weaker axioms for his classes and proves some of the following lemmas only using these weaker axioms. However the classes we are interested in do satisfy the above (stronger) requirements. One should think of a Serre class as a class of groups we want to *ignore*.

Example 2.0.4. We give three Serre classes without proof.

- The class $\mathcal{C} = \{0\}$. With this class the following Hurewicz and Whitehead theorem will just restate be the classical theorems.
- The class $\mathcal{C}$ of all torsion groups. Using this class we can prove the rational version of the Hurewicz and Whitehead theorems.
- The class $\mathcal{C}$ of all uniquely divisible groups. Note that these groups can be given a unique $\mathbb{Q}$-vector space structure (and conversely every $\mathbb{Q}$-vector space is uniquely divisible).

The first three axioms of Serre classes are easily checked. For the group homology we find a calculation of the group homology of cyclic groups in [Moe08]. The group homology itself is also a torsion group, this result extends to all torsion groups. As noted by Hilton in [Hil04] we think of Serre classes as a generalized 0. This means that we can also express some kind of generalized injective and surjectivity. Here we only need the notion of a $\mathcal{C}$-isomorphism:

Definition 2.0.5. Let $\mathcal{C}$ be a Serre class and let $f : A \to B$ be a map of abelian groups. Then f is a $\mathcal{C}$*-isomorphism* if both the kernel and cokernel lie in $\mathcal{C}$.

Note that the maps $0 \to C$ and $C \to 0$ are $\mathcal{C}$-isomorphisms for any $C \in \mathcal{C}$. More importantly the 5-lemma also holds for $\mathcal{C}$-isos and we have the 2-out-of-3 property: whenever f, g and $g \circ f$ are maps such that two of them are $\mathcal{C}$-iso, then so is the third.

In the following arguments we will consider fibrations and need to compute homology thereof. Unfortunately there is no long exact sequence for homology of a fibration, however the following lemma expresses something similar. It is usually proven with spectral sequences, [Ser53, Ch. 2 Thm 1]. However in [KK04] we find a more geometric proof for rational coefficients, which we generalize here to Serre classes.

Lemma 2.0.6. *Let $\mathcal{C}$ be a Serre class and $p : E \twoheadrightarrow B$ be a fibration between 0-connected spaces with a 0-connected fiber F. If $\widetilde{H}_i(F) \in \mathcal{C}$ for all $i < n$ and B is m-connected, then*

- *$H_i(E, F) \to H_i(B, b_0)$ is a $\mathcal{C}$-iso for $i \leq n + m$ and*
- *$H_i(E) \to H_i(B)$ is a $\mathcal{C}$-iso for all $i < n + m$.*

Proof. We will first replace the fibration by a fiber bundle. This is done by going to simplicial sets and replace the induced map by a minimal fibration as follows. The fibration p induces a fibration $S(E) \xrightarrow{S(p)} S(B)$, which can be factored as $S(E) \xrightarrow{\simeq} M \twoheadrightarrow S(B)$, where the map $M \twoheadrightarrow S(B)$ is minimal and hence a fiber bundle [JT99]. By realizing we obtain the following diagram:

$$\begin{tikzcd} |M| \arrow[d, two heads] & |S(E)| \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, two heads] & E \arrow[d, two heads] \\ |S(B)| & |S(B)| \arrow[l, "\mathbf{id}"] \arrow[r, "\simeq"] & B \end{tikzcd}$$

The fibers of all fibrations are weakly equivalent by the long exact sequence, so the assumptions of the lemma also hold for the fiber bundle. To prove the lemma, it is enough to do so for the fiber bundle $|M| \twoheadrightarrow |S(B)|$.

So we can assume E and B to be a CW complexes and $E \twoheadrightarrow B$ to be a fiber bundle. We will do induction on the skeleton B^k. By connectedness we can assume $B^0 = \{b_0\}$. Restrict E to B^k and note $E^0 = F$. Now the base case is clear: $H_i(E^0, F) \to H_i(B^0, b_0)$ is a $\mathcal{C}$-iso.

For the induction step, consider the long exact sequence in homology for the triples (E^{k+1}, E^k, F) and (B^{k+1}, B^k, b_0):

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}(E^{k+1}, E^k) \arrow[r] \arrow[d] & H_i(E^k, F) \arrow[r] \arrow[d] & H_i(E^{k+1}, F) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{i+1}(B^{k+1}, B^k) \arrow[r] & H_i(B^k, b_0) \arrow[r] & H_i(B^{k+1}, b_0) \arrow[r] & \cdots
\end{tikzcd}$$

The morphism in the middle is a $\mathcal{C}$-iso by induction. We will prove that the left morphism is a $\mathcal{C}$-iso which implies by the five lemma that the right morphism is one as well.

As we are working with relative homology $H_{i+1}(E^{k+1}, E^k)$, we only have to consider the interiors of the $(k+1)$-cells (by excision). Each interior of a $(k+1)$-cell is a product, as p is a fiber bundle. So we note that we have an isomorphism:

$$H_{i+1}(E^{k+1}, E^k) \cong H_{i+1}(\coprod_\alpha D_\alpha^{k+1} \times F, \coprod_\alpha S_\alpha^k \times F).$$

Now we can apply the Künneth theorem for this product to obtain a natural short exact sequence, furthermore we apply the Künneth theorem for $(B^{k+1}, B^k) \times *$ to obtain a second short exact sequence as follows.

$$\begin{tikzcd}
0 \arrow[r] & (H(B^{k+1}, B^k) \otimes H(F))_{i+1} \arrow[r] \arrow[d, "p'"] & H_{i+1}(E^{k+1}, E^k) \arrow[r] \arrow[d, "p_*"] & \mathrm{Tor}(H(B^{k+1}, B^k), H(F))_i \arrow[r] \arrow[d, "p''"] & 0 \\
0 \arrow[r] & H_{i+1}(B^{k+1}, B^k) \arrow[r] & H_{i+1}(B^{k+1}, B^k) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Now it remains to show that p' and p'' are $\mathcal{C}$-iso, as it will then follow from the five lemma that p_* is a $\mathcal{C}$-iso.

First note that p' is surjective as it is an isomorphism on the subspace $H_{i+1}(B^{k+1}, B^k) \otimes H_0(F)$. Its kernel on the other hand is precisely given by the terms $H_{i+1-q}(B^{k+1}, B^k) \otimes H_q(F)$ for $q > 0$. By assumption we have $H_q(F) \in \mathcal{C}$ for all $0 < q < n$ and $H_{i+1}(B^{k+1}, B^k) = 0$ for all $i + 1 \leq m$. By the tensor axiom of a Serre class the kernel is in $\mathcal{C}$ for all $i < n + m$. So indeed p' is a $\mathcal{C}$-iso for all $i < n + m$.

For p'' a similar reasoning holds, it is clearly surjective and we only need to prove that the kernel of p'' (which is the Tor group itself) is in $\mathcal{C}$. First notice that $\mathrm{Tor}(H_i(B^{k+1}, B^k), H_0(F)) = 0$ as $H_0(F) \cong \mathbb{Z}$. Then consider the other terms of the graded Tor group. Again we use the assumed bounds to conclude that the Tor group is in $\mathcal{C}$ for $i \leq n + m$. So indeed p'' is a $\mathcal{C}$-iso for all $i \leq n + m$.

Now we conclude that $p_* : H_{i+1}(B^{k+1}, B^k) \to H_{i+1}(E^{k+1}, E^k)$ is indeed a $\mathcal{C}$-iso for all $i < n + m$. And by the long exact sequence of triples shown above we get a $\mathcal{C}$-iso $p_* : H_i(E^{k+1}, F) \to H_i(B^{k+1}, b_0)$ for all $i \leq n + m$. This finished the induction on k.

This concludes that $H_i(E, F) \to H_i(B, b_0)$ is a $\mathcal{C}$-iso and by another application of the long exact sequence (of the pair (E, F)) and the five lemma we get the $\mathcal{C}$-iso $H_i(E) \to H_i(B)$. $\square$

Lemma 2.0.7. *Let $\mathcal{C}$ be a Serre class and $G \in \mathcal{C}$. Then for all $n > 0$ and all $i > 0$ we have $H_i(K(G, n)) \in \mathcal{C}$.*

Proof. We prove this by induction on n. The base case $n = 1$ follows from group homology. By considering the nerve of G we can construct $K(G, 1)$. This construction can be related to the bar construction as found in [Moe08]. This then identifies the homology of the Eilenberg-MacLane space with the group homology which is in $\mathcal{C}$ by the axioms:

$$H_i(K(G, 1); \mathbb{Z}) \cong H_i(G; \mathbb{Z}) \in \mathcal{C}.$$

Suppose we have proven the statement for n. If we consider the case of $n + 1$ we can use the path fibration to relate it to the case of n:

$$\Omega K(G, n + 1) \to PK(G, n + 1) \twoheadrightarrow K(G, n + 1)$$

Now $\Omega K(G, n + 1) = K(G, n)$, and we can apply Lemma 2.0.6 as the reduced homology of the fiber is in $\mathcal{C}$ by induction hypothesis. Conclude that the homology of $PK(G, n + 1)$ is $\mathcal{C}$-isomorphic to the homology of $K(G, n)$. Since $\widetilde{H}_*(PK(G, n + 1)) = 0$, we get $\widetilde{H}_*(K(G, n + 1)) \in \mathcal{C}$. $\square$

For the main theorem we need the following decomposition of spaces. The construction can be found in [Ser53] or [KK04].

Lemma 2.0.8. *(Whitehead tower) We can decompose a 0-connected space X into fibrations:*

$$\cdots \twoheadrightarrow X(n + 1) \twoheadrightarrow X(n) \twoheadrightarrow X(n - 1) \twoheadrightarrow \cdots \twoheadrightarrow X(1) = X,$$

such that:

- *$K(\pi_n(X), n-1) \hookrightarrow X(n+1) \twoheadrightarrow X(n)$ is a fiber sequence.*
- *There is a space X'_n weakly equivalent to $X(n)$ such that $X(n+1) \hookrightarrow X'_n \twoheadrightarrow K(\pi_n(X), n)$ is a fiber sequence.*
- *$X(n)$ is $(n-1)$-connected and $\pi_i(X(n)) \to \pi_i(X)$ is an isomorphism for all $i \geq n$.*

Theorem 2.0.9. *(Absolute Serre-Hurewicz Theorem) Let $\mathcal{C}$ be a Serre class. Let X a 1-connected space. If $\pi_i(X) \in C$ for all $i < n$, then $H_i(X) \in C$ for all $i < n$ and the Hurewicz map $h : \pi_i(X) \to H_i(X)$ is a $\mathcal{C}$-isomorphism for all $i \leq n$.*

Proof. We will prove the lemma by induction on n. Note that the base case ($n = 1$) follows from the 1-connectedness.

For the induction step we may assume that $H_i(X) \in \mathcal{C}$ for all $i < n-1$ and that $h_{n-1} : \pi_{n-1}(X) \to H_{n-1}(X)$ is a $\mathcal{C}$-iso by induction hypothesis. Furthermore the theorem assumes that $\pi_{n-1}(X) \in \mathcal{C}$ and hence we conclude $H_{n-1}(X) \in \mathcal{C}$.

It remains to show that h_n is a $\mathcal{C}$-iso. Use the Whitehead tower from Lemma 2.0.8 to obtain $\cdots \twoheadrightarrow X(3) \twoheadrightarrow X(2) = X$. Note that each $X(j)$ is 1-connected and that $X(2) = X(1) = X$.

Claim 2.0.10. *For all $j < n$ and $i \leq n$ the induced map $H_i(X(j+1)) \to H_i(X(j))$ is a $\mathcal{C}$-iso.*

Note that $X(j+1) \twoheadrightarrow X(j)$ is a fibration with $F = K(\pi_j(X), j-1)$ as its fiber. So by Lemma 2.0.7 we know $H_i(F) \in \mathcal{C}$ for all i. Apply Lemma 2.0.6 to obtain a $\mathcal{C}$-iso $H_i(X(j+1)) \to H_i(X(j))$ for all $j < n$ and all $i > 0$. This proves the claim.

Considering this claim for all $j < n$ gives a chain of $\mathcal{C}$-isos $H_i(X(n)) \to H_i(X(n-1)) \to \cdot \to H_i(X(2)) = H_i(X)$ for all $i \leq n$. Consider the following diagram:

$$\begin{tikzcd}
\pi_n(X(n)) \arrow[r, "\cong"] \arrow[d, "\cong"] & H_n(X(n)) \arrow[d, "\mathcal{C}-iso"] \\
\pi_n(X) \arrow[r] & H_n(X)
\end{tikzcd}$$

where the map on the top is an isomorphism by the classical Hurewicz theorem (and $X(n)$ is $(n-1)$-connected), the map on the left is an isomorphism by the Whitehead tower and the map on the right is a $\mathcal{C}$-iso by the claim.

It follows that the bottom map is a $\mathcal{C}$-iso. □

Theorem 2.0.11. *(Relative Serre-Hurewicz Theorem) Let $\mathcal{C}$ be a Serre class. Let $A \subset X$ be 1-connected spaces such that $\pi_2(A) \to \pi_2(X)$*

is surjective. If $\pi_i(X, A) \in \mathcal{C}$ for all $i < n$, then $H_i(X, A) \in \mathcal{C}$ for all $i < n$ and the Hurewicz map $h : \pi_i(X, A) \to H_i(X, A)$ is a $\mathcal{C}$-isomorphism for all $i \leq n$.

Proof. Note that we can assume $A \neq \emptyset$. We will prove by induction on n, the base case again follows by 1-connectedness.

Let PX be that path space on X and $Y \subset PX$ be the subspace of paths of which the endpoint lies in A. Now we get a fibration (of pairs) by sending the path to its endpoint:

$$p : (PX, Y) \twoheadrightarrow (X, A),$$

with ΩX as its fiber. We get long exact sequences of homotopy groups of the triples $\Omega X \subset Y \subset PX$ and $* \in A \subset X$:

$$\begin{array}{ccccccccc} \pi_i(Y, PX) & \to & \pi_i(PX, \Omega X) & \to & \pi_i(PX, Y) & \to & \pi_{i-1}(Y, \Omega X) & \to & \pi_{i-1}(PX, \Omega X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_i(A, *) & \longrightarrow & \pi_i(X, *) & \longrightarrow & \pi_i(X, A) & \longrightarrow & \pi_{i-1}(A, *) & \longrightarrow & \pi_{i-1}(X, *) \end{array}$$

The outer vertical maps are isomorphisms (again by a long exact sequence argument), hence the center vertical map is an isomorphism. Furthermore $\pi_i(PX) = 0$ as it is a path space, hence $\pi_{i-1}(Y) \cong \pi_i(PX, Y) \cong \pi_i(X, A)$. By assumption we have $\pi_1(X, A) = \pi_2(X, A) = 0$. So Y is 1-connected. Furthermore $\pi_{i-1}(Y) \in \mathcal{C}$ for all $i < n$.

Now we can use the previous Serre-Hurewicz theorem to conclude $H_{i-1}(Y) \in \mathcal{C}$ for all $i < n$ and $\pi_{n-1}(Y) \xrightarrow{h} H_{n-1}(Y)$ is an $\mathcal{C}$-iso. We are in the following situation:

$$\begin{tikzcd} \pi_{n-1}(Y) \arrow[d, "\mathcal{C}\text{-iso}"] & \pi_n(PX, Y) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \pi_n(X, A) \arrow[d] \\ H_{n-1}(Y) & H_n(PX, Y) \arrow[l, "\cong"'] \arrow[r, "\mathcal{C}\text{-iso}"] & H_n(X, A) \end{tikzcd}$$

The horizontal maps on the left are isomorphisms by long exact sequences, this gives us that the middle vertical map is a $\mathcal{C}$-iso. The horizontal maps on the right are $\mathcal{C}$-isos by the above and a relative version of Lemma 2.0.6. Now we conclude that $\pi_n(X, A) \to H_n(X, A)$ is also a $\mathcal{C}$-iso and that $H_i(X, A) \in \mathcal{C}$ for all $i < n$. □

Theorem 2.0.12. *(Serre-Whitehead Theorem) Let $\mathcal{C}$ be a Serre class. Let $f : X \to Y$ be a map between 1-connected spaces such that $\pi_2(f)$ is surjective. Then $\pi_i(f)$ is a C-iso for all $i < n \iff H_i(f)$ is a $\mathcal{C}$-iso for all $i < n$.*

Proof. Consider the mapping cylinder B_f of f, i.e. factor the map f as a cofibration followed by a trivial fibration $f : A \hookrightarrow B_f \twoheadrightarrow B$. The inclusion $A \subset B_f$ gives a long exact sequence of homotopy groups and homology groups:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{i+1}(B_f, A) \arrow[r] \arrow[d] & \pi_i(A) \arrow[r, "f_*"] \arrow[d] & \pi_i(B) \arrow[r] \arrow[d] & \pi_i(B_f, A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{i+1}(B_f, A) \arrow[r] & H_i(A) \arrow[r, "f_*"] & H_i(B) \arrow[r] & H_i(B_f, A) \arrow[r] & \cdots
\end{tikzcd}$$

We now have the equivalence of the following statements:

1. $\pi_i(f)$ is a $\mathcal{C}$-iso for all $i < n$
2. $\pi_i(B_f, A) \in \mathcal{C}$ for all $i < n$
3. $H_i(B_f, A) \in \mathcal{C}$ for all $i < n$
4. $H_i(f)$ is a $\mathcal{C}$-iso for all $i < n$.

Where (1) $\iff$ (2) and (3) $\iff$ (4) hold by exactness and (2) $\iff$ (3) by the Serre-Hurewicz theorem. □

2.1 FOR RATIONAL EQUIVALENCES

Lemma 2.1.1. *Let $\mathcal{C}$ be the Serre class of all torsion groups. Then f is a $\mathcal{C}$-iso $\iff$ $f \otimes \mathbb{Q}$ is an isomorphism.*

Proof. First note that if $C \in \mathcal{C}$ then $C \otimes \mathbb{Q} = 0$.

Then consider the exact sequence

$$0 \to \ker(f) \to A \xrightarrow{f} B \to \operatorname{coker}(f) \to 0$$

and tensor this sequence with $\mathbb{Q}$. In this tensored sequence the kernel and cokernel vanish if and only if $f \otimes \mathbb{Q}$ is an isomorphism. □

Combining this lemma and Theorem 2.0.9 we get the following corollary for rational homotopy theory:

Corollary 2.1.2. *(Rational Hurewicz Theorem) Let X be a 1-connected space. If $\pi_i(X) \otimes \mathbb{Q} = 0$ for all $i < n$, then $H_i(X; \mathbb{Q}) = 0$ for all $i < n$. Furthermore we have an isomorphism for all $i \leq n$:*

$$\pi_i(X) \otimes \mathbb{Q} \xrightarrow{\cong} H_i(X; \mathbb{Q})$$

By using the class of $\mathbb{Q}$-vector spaces we get a dual theorem.

Corollary 2.1.3. *(Rational Hurewicz Theorem 2) Let X be a 1-connected space. The homotopy groups $\pi_i(X)$ are $\mathbb{Q}$-vector spaces for all $i > 0$ if and only if $H_i(X)$ are $\mathbb{Q}$-vector spaces for all $i > 0$.*

Theorem 2.0.12 also applies verbatim to rational homotopy theory. However we would like to avoid the assumption that $\pi_2(f)$ is surjective. In [FHT01] we find a way to work around this.

Corollary 2.1.4. *(Rational Whitehead Theorem) Let $f : X \to Y$ be a map between 1-connected spaces. Then f is a rational equivalence $\iff H_*(f;\mathbb{Q})$ is an isomorphism.*

Proof. We will replace f by some map f_1 which is surjective on π_2. First consider $\Gamma = \pi_2(Y)/\operatorname{im}(\pi_2(f))$ and its Eilenberg-MacLane space $K = K(\Gamma, 2)$. There is a map $q : Y \to K$ inducing the projection map $\pi_2(q) : \pi_2(Y) \to \Gamma$.

We can factor q as

$$\begin{tikzcd} Y \arrow[rr, "\lambda", "\simeq"'] \arrow[dr, "q"'] & & Y \times_K MK \arrow[dl, two heads, "\overline{q}"] \\ & K & \end{tikzcd}$$

Where MK is the Moore path space and $\overline{q}$ is induced by the map sending a path to its endpoint. Now $\overline{q}\lambda f$ is homotopic to the constant map, so there is a homotopy $h : \overline{q}\lambda f \sim *$ which we can lift against the fibration $\overline{q}$ to a homotopy $h' : \lambda f \sim f_1$ with $\overline{q} f_1 = *$. In other words f_1 lands in the fiber of $\overline{q}$.

We get a commuting square when applying π_2:

$$\begin{tikzcd} \pi_2(X) \arrow[r, "\pi_2(f_1)"] \arrow[d, "\pi_2(f)"] & \pi_2(Y \times_K PK) \arrow[d, "\pi_2(i)"] \\ \pi_2(Y) \arrow[r, "\cong"] & \pi_2(Y \times_K MK) \end{tikzcd}$$

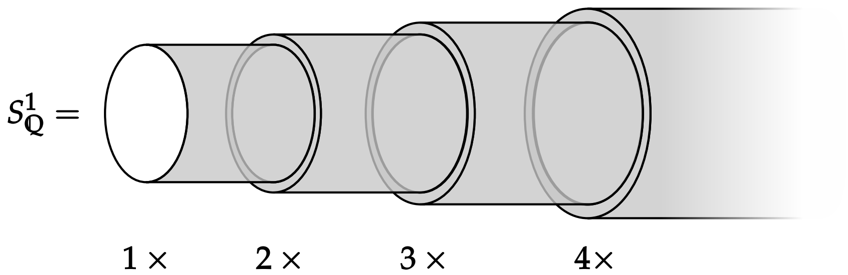
The important observation is that by the long exact sequence $\pi_*(i) \otimes \mathbb{Q}$ and $H_*(i;\mathbb{Q})$ are isomorphisms (here we use that $\Gamma \otimes \mathbb{Q} = 0$ and that tensoring with $\mathbb{Q}$ is exact). So by the above square $\pi_*(f_1) \otimes \mathbb{Q}$ is an isomorphism if and only if $\pi_*(f) \otimes \mathbb{Q}$ is (and similarly for homology). Finally we note that $\pi_2(f_1)$ is surjective, so Theorem 2.0.12 applies and the result also holds for f. $\square$

3 RATIONALIZATIONS

In this section we will prove the existence of rationalizations $X \to X_{\mathbb{Q}}$. We will do this in a cellular way. The n-spheres play an important role here, so their rationalizations will be discussed first. In this section 1-connectedness of spaces will play an important role.

3.1 RATIONALIZATION OF S^n

We will construct $S^n_{\mathbb{Q}}$ as an infinite telescope, as depicted for $n = 1$ in the following picture.



The space will consist of multiple copies of S^n, one for each $k \in \mathbb{N}^{>0}$, glued together by $(n+1)$-cells. The role of the kth copy (together with the gluing) is to be able to "divide by k".

So $S^n_{\mathbb{Q}}$ will be of the form $S^n_{\mathbb{Q}} = \bigvee_{k>0} S^n \cup_h \coprod_{k>0} D^{n+1}$. We will define the attaching map h by doing the construction in stages.

We start the construction with $S^n(1) = S^n$. Now assume $S^n(r) = \bigvee_{i=1}^{r} S^n \cup_{h(r)} \coprod_{i=1}^{r-1} D^{n+1}$ is constructed. Let $i : S^n \to S^n(r)$ be the inclusion into the last (i.e. rth) sphere, and let $g : S^n \to S^n$ be a representative for the class $(r+1)[\mathbf{id}] \in \pi_n(S^n)$. Combine the two maps to obtain $\phi : S^n \to S^n \vee S^n \xrightarrow{i \vee g} S^n(r) \vee S^n$. We define $S^n(r+1)$ as the pushout:

$$\begin{tikzcd}
S^n \arrow[r, "\phi"] \arrow[d, hook] & S^n(r) \vee S^n \arrow[d] \\
D^{n+1} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & S^n(r+1)
\end{tikzcd}$$

So that $S^n(r+1) = \bigvee_{i=1}^{r+1} S^n \cup_{h(r+1)} \coprod_{i=1}^{r} D^{n+1}$. To finish the construction we define $S^n_{\mathbb{Q}} = \operatorname{colim}_r S^n(r)$.

We note two things here. First, at any stage, the inclusion $i : S^n \to S^n(r)$ into the rth sphere is a weak equivalence, as we can collapse the (finite) telescope to the last sphere. This identifies $\pi_n(S^n(r)) = \mathbb{Z}$ for all r. Secondly, let $i_r : S^n \to S^n(r+1)$ be the inclusion of the rth sphere and let $i_{r+1} : S^n \to S^n(r+1)$ be the inclusion of the last sphere, then $[i_r] = (r+1)[i_{r+1}] \in \pi^n(S^n(r+1))$, by construction. This means that we can divide the class $[i_r]$ by $r+1$. This shows that the inclusion $S^n(r) \to S^n(r+1)$ induces a multiplication by $r+1$ under the identification $\pi^n(S^n(r)) = \mathbb{Z}$ for all r.

The nth homotopy group of $S^n_{\mathbb{Q}}$ can be calculated as follows. We use the fact that the homotopy groups commute with filtered colimits [May99, 9.4] to compute $\pi_n(S^n_{\mathbb{Q}})$ as the colimit of the terms $\pi_n(S^n(r)) \cong \mathbb{Z}$ and the induced maps as depicted in the following diagram:

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\times 4} \mathbb{Z} \dashrightarrow \mathbb{Q}$$

Moreover we note that the generator $1 \in \mathbb{Z} = \pi_n(S^n)$ is sent to $1 \in \mathbb{Q} = \pi_n(S^n_{\mathbb{Q}})$ via the inclusion $S^n \to S^n_{\mathbb{Q}}$ of the initial sphere. However the other homotopy groups are harder to calculate as we have generally no idea what the induced maps are. But in the case of $n = 1$, the other homotopy groups of S^1 are trivial.

Corollary 3.1.1. *The inclusion $S^1 \to S^1_{\mathbb{Q}}$ is a rationalization.*

For $n > 1$ we can resort to homology, which also commutes with filtered colimits [May99, 14.6]. By connectedness we have $H_0(S^n_{\mathbb{Q}}) = \mathbb{Z}$ and for $i \neq 0, n$ we have $H_i(S^n) = 0$, so the colimit is also trivial. For $i = n$ we can use the same sequence as above (or use the Hurewicz theorem) to conclude:

$$H_i(S^n_{\mathbb{Q}}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0 \\ \mathbb{Q}, & \text{if } i = n \\ 0, & \text{otherwise.} \end{cases}$$

By the Serre-Hurewicz theorem (Corollary 2.1.3) we see that $S^n_{\mathbb{Q}}$ is indeed rational. Then by the Serre-Whitehead theorem (Corollary 2.1.4) the inclusion map $S^n \to S^n_{\mathbb{Q}}$ is a rationalization.

Corollary 3.1.2. *The inclusion $S^n \to S^n_{\mathbb{Q}}$ is a rationalization.*

The *rational disk* is now defined as cone of the rational sphere: $D^{n+1}_{\mathbb{Q}} = CS^n_{\mathbb{Q}}$. By the naturality of the cone construction we get the following commutative diagram of inclusions.

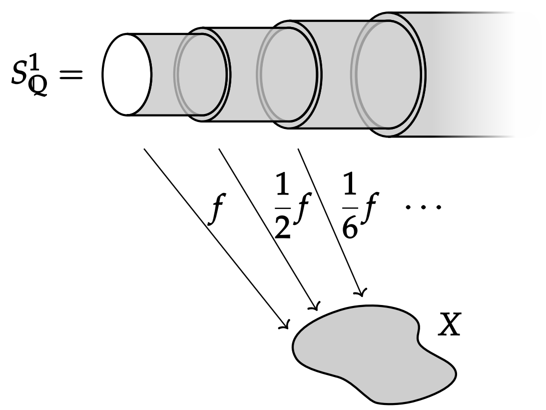
$$\begin{tikzcd} S^n \arrow[r, hook] \arrow[d, hook] & S^n_{\mathbb{Q}} \arrow[d, hook] \\ D^{n+1} \arrow[r, hook] & D^{n+1}_{\mathbb{Q}} \end{tikzcd}$$

Lemma 3.1.3. *Let X be a rational space and $f : S^n \to X$ be a map. Then this map extends to a map $f' : S^n_{\mathbb{Q}} \to X$ making the following diagram commute.*

$$\begin{tikzcd} S^n \arrow[r, "i"] \arrow[dr, "f"'] & S^n_{\mathbb{Q}} \arrow[d, dashed, "f'"] \\ & X \end{tikzcd}$$

Furthermore f' is determined up to homotopy (i.e. any map f'' with $f''i = f$ is homotopic to f') and homotopic maps have homotopic extensions (i.e. if $f \simeq g$, then $f' \simeq g'$).

Proof. Note that f represents a class $\alpha \in \pi_n(X)$. Since $\pi_n(X)$ is a $\mathbb{Q}$-vector space there exists elements $\frac{1}{2}\alpha, \frac{1}{3}\alpha, \ldots$ with representatives $\frac{1}{2}f, \frac{1}{3}f, \ldots$. Recall that $S^n_{\mathbb{Q}}$ consists of many copies of S^n, we can define f' on the kth copy to be $\frac{1}{k!}f$, as depicted in the following diagram.



Since $[\frac{1}{(k-1)!}f] = k[\frac{1}{k!}f] \in \pi_n(X)$ we can define f' accordingly on the $n+1$-cells. Since our inclusion $i : S^n \hookrightarrow S^n_{\mathbb{Q}}$ is in the first sphere, we get $f = f' \circ i$.

Let f'' be any map such that $f''i = f$. Then f'' also represents α and all the functions $\frac{1}{2}f''$, $\frac{1}{6}f''$,... are hence homotopic to $\frac{1}{2}f$, $\frac{1}{6}f$,.... So indeed f is homotopic to f''.

Now if g is homotopic to f. We can extend the homotopy h in a similar way to the rational sphere. Hence the extensions are homotopic. □

3.2 RATIONALIZATIONS OF ARBITRARY SPACES

Having rational cells we wish to replace the cells in a CW complex X by the rational cells to obtain a rationalization.

Lemma 3.2.1. *Any simply connected CW complex admits a rationalization.*

Proof. Let X be a CW complex. We will define $X_{\mathbb{Q}}$ with induction on the skeleton. Since X is simply connected we can start with $X^0_{\mathbb{Q}} = X^1_{\mathbb{Q}} = *$. Now assume that the rationalization $X^k \xrightarrow{\phi^k} X^k_{\mathbb{Q}}$ is already defined. Let A be the set of $k+1$-cells and $f_\alpha : S^k \to X^{k+1}$ be the attaching maps. Then by Lemma 3.1.3 these extend to $g_\alpha = (\phi^k \circ f_\alpha)' : S^k_{\mathbb{Q}} \to X^k_{\mathbb{Q}}$. This defines $X^{k+1}_{\mathbb{Q}}$ as the pushout in the following diagram.

$$\begin{tikzcd}
\coprod_A S^n_{\mathbb{Q}} \arrow[r, "(g_\alpha)"] \arrow[d, hook] & X^k_{\mathbb{Q}} \arrow[d, dashed] \\
\coprod_A D^{n+1}_{\mathbb{Q}} \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner"] & X^{k+1}_{\mathbb{Q}}
\end{tikzcd}$$

Now by the universal property of X^{k+1}, we get a map $\phi^{k+1} : X^{k+1} \to X^{k+1}_{\mathbb{Q}}$ which is compatible with ϕ^k and which is a rationalization. $\square$

Lemma 3.2.2. *Any simply connected space admits a rationalization.*

Proof. Let $Y \xrightarrow{f} X$ be a CW approximation and let $Y \xrightarrow{\phi} Y_{\mathbb{Q}}$ be the rationalization of Y. Now we define $X_{\mathbb{Q}}$ as the double mapping cylinder (or homotopy pushout):

$$X_{\mathbb{Q}} = X \cup_f (Y \times I) \cup_\phi Y_{\mathbb{Q}}.$$

with the obvious inclusion $\psi : X \to X_{\mathbb{Q}}$. By excision we see that $H_*(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \cong H_*(X \cup_f (Y \times I), Y \times 1) = 0$. So by the long exact sequence of the inclusion we get $H_*(X_{\mathbb{Q}}) \cong H_*(Y_{\mathbb{Q}})$, which proves by the rational Hurewicz theorem that $X_{\mathbb{Q}}$ is a rational space. At last we note that $H_*(X_{\mathbb{Q}}, X; \mathbb{Q}) \cong H_*(Y_{\mathbb{Q}}, Y; \mathbb{Q}) = 0$, since ϕ was a rationalization. This proves that $H_*(\psi; \mathbb{Q})$ is an isomorphism, so by the rational Whitehead theorem, ψ is a rationalization. $\square$

Theorem 3.2.3. *The above construction is in fact a* localization*, i.e. for any map $f : X \to Z$ to a rational space Z, there is an extension $f' : X_{\mathbb{Q}} \to Z$ making the following diagram commute.*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & X_{\mathbb{Q}} \arrow[d, dashed, "f'"] \\ & Z \end{tikzcd}$$

Moreover, f' is determined up to homotopy and homotopic maps have homotopic extensions.

We will note prove that above theorem (it is analogue to Lemma 3.1.3), but refer to [FHT01]. The extension property allows us to define a rationalization of maps. Given $f : X \to Y$, we can consider the composite $if : X \to Y \to Y_{\mathbb{Q}}$. Now this extends to $(if)' : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$. Note that this construction is not functorial, since there are choices of homotopies involved. When passing to the homotopy category, however, this construction *is* functorial and has an universal property.

We already mentioned in the first section that for rational spaces the notions of weak equivalence and rational equivalence coincide. Now that we always have a rationalization we have:

Corollary 3.2.4. *Let $f : X \to Y$ be a map, then f is a rational equivalence if and only if $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is a weak equivalence.*

Corollary 3.2.5. *The homotopy category of 1-connected rational spaces is equivalent to the rational homotopy category of 1-connected spaces.*

3.3 OTHER CONSTRUCTIONS

There are others ways to obtain a rationalization. One of them relies on the observations that it is easy to rationalize Eilenberg-MacLane spaces. Since we already have a rationalization at hand the details in this section will be skipped and the focus lies on the construction.

Remark 3.3.1. *Let A be an abelian group and $n \geq 1$. Then*

$$K(A, n) \to K(A \otimes \mathbb{Q}, n)$$

is a rationalization

Any simply connected space can be decomposed into a Postnikov tower $X \to \ldots \twoheadrightarrow P_2(X) \twoheadrightarrow P_1(X) \twoheadrightarrow P_0(X)$ [May99, Chapter 22.4]. Furthermore if X is a simply connected CW complex, $P_n(X)$ can be constructed from $P_{n-1}(X)$ as the pullback in

$$\begin{tikzcd}
P_n(X) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & PK(\pi_n(X), n+1) \arrow[d, two heads] \\
P_{n-1}(X) \arrow[r, "k_{n-1}"] & K(\pi_n(X), n+1),
\end{tikzcd}$$

where the map k_{n-1} is called the k-invariant. We will only need its existence for the construction. The rationalization can now be constructed with induction on this Postnikov tower. Start the induction with $X_{\mathbb{Q}}(2) = K(\pi_2(X) \otimes \mathbb{Q}, 2)$. Now assume we constructed $X_{\mathbb{Q}}(r-1)$ compatible with the k-invariant described above. We are in the following situation:

$$\begin{tikzcd}
P_r(X) \arrow[rr] \arrow[dd, two heads] & & PK(\pi_r(X), r+1) \arrow[dr] \arrow[dd, two heads] & \\
 & & & PK(\pi_r(X) \otimes \mathbb{Q}, r+1) \arrow[dd, two heads] \\
P_{r-1}(X) \arrow[rr, "k_{r-1}"] \arrow[dr, "\phi_{r-1}"] & & K(\pi_r(X), r+1) \arrow[dr] & \\
 & X_{\mathbb{Q}}(r-1) \arrow[rr] & & K(\pi_r(X) \otimes \mathbb{Q}, r+1)
\end{tikzcd}$$

where the bottom square is our induction hypothesis, the right square is by naturality of the path space fibration and the back face is the pullback described above. We can define $X_{\mathbb{Q}}(r)$ to be the pullback of the front face, which induces a map $\phi_r : P_r(X) \to X_{\mathbb{Q}}(r)$. By inspecting the long exact sequence of the fibration $X_{\mathbb{Q}}(r) \twoheadrightarrow X_{\mathbb{Q}}(r-1)$ we see that ϕ_r is indeed a rationalization.

We finish the construction by defining $X_{\mathbb{Q}} = \lim_r X_{\mathbb{Q}}(r)$. For more details, one can read [SR05] or [Ber12].

Part II

CDGA'S AS ALGEBRAIC MODELS

4 HOMOTOPY THEORY FOR CDGA'S

Recall that a cdga A is a commutative differential graded algebra, meaning that

- it has a grading: $A = \bigoplus_{n \in \mathbb{N}} A^n$,
- it has a differential: $d : A \to A$ with $d^2 = 0$,
- it has a multiplication: $\mu : A \otimes A \to A$ which is associative and unital and
- it is commutative: $xy = (-1)^{|x| \cdot |y|} yx$.

And all of the above structure is compatible with each other (e.g. the differential is a derivation of degree 1, the maps are graded, ...). The exact requirements are stated in the appendix on algebra. An algebra A is augmented if it has a specified map (of algebras) $A \xrightarrow{\epsilon} \mathbb{k}$. Furthermore we adopt the notation $A^{\leq n} = \bigoplus_{k \leq n} A^k$ and similarly for $\geq n$.

There is a left adjoint Λ to the forgetful functor U which assigns the free graded commutative algebras ΛV to a graded module V. This extends to an adjunction (also called Λ and U) between commutative differential graded algebras and differential graded modules. We denote the subspace of elements of wordlength n by $\Lambda^n V$ (note that this has nothing to do with the grading on V).

In homological algebra we are especially interested in *quasi isomorphisms*, i.e. maps $f : A \to B$ inducing an isomorphism on cohomology: $H(f) : HA \cong HB$. This notions makes sense for any object with a differential.

We furthermore have the following categorical properties of cdga's:

- The finite coproduct in $\mathbf{CDGA}_{\mathbb{k}}$ is the (graded) tensor product.
- The finite product in $\mathbf{CDGA}_{\mathbb{k}}$ is the cartesian product (with pointwise operations).
- The equalizer (resp. coequalizer) of f and g is given by the kernel (resp. cokernel) of $f - g$. Together with the (co)products this defines pullbacks and pushouts.
- $\mathbb{k}$ and 0 are the initial and final object.

4.1 COCHAIN MODELS FOR THE n-DISK AND n-SPHERE

We will first define some basic cochain complexes which model the n-disk and n-sphere. $D(n)$ is the cochain complex generated by one element $b \in D(n)^n$ and its differential $c = d(b) \in D(n)^{n+1}$. On the other hand we define $S(n)$ to be the cochain complex generated by one element $a \in S(n)^n$ with trivial differential (i.e. $da = 0$). In other words:

$$D(n) = \ldots \to 0 \to \Bbbk \to \Bbbk \to 0 \to \ldots$$

$$S(n) = \ldots \to 0 \to \Bbbk \to 0 \to 0 \to \ldots$$

Note that $D(n)$ is acyclic for all n, or put in different words: $j_n : 0 \to D(n)$ induces an isomorphism in cohomology. The sphere $S(n)$ has exactly one non-trivial cohomology group $H^n(S(n)) = \Bbbk \cdot [a]$. There is an injective function $i_n : S(n+1) \to D(n)$, sending a to c. The maps j_n and i_n play the following important role in the model structure of cochain complexes, where weak equivalences are quasi isomorphisms, fibrations are degreewise surjective and cofibrations are degreewise injective for positive degrees [GS06, Example 1.6].

Claim 4.1.1. *The set $I = \{i_n : S(n+1) \to D(n) \mid n \in \mathbb{N}\}$ generates all cofibrations and the set $J = \{j_n : 0 \to D(n) \mid n \in \mathbb{N}\}$ generates all trivial cofibrations.*

As we do not directly need this claim, we omit the proof. However, in the next section we will prove a similar result for cdga's in detail.

$S(n)$ plays a another special role: maps from $S(n)$ to some cochain complex X correspond directly to elements in the kernel of $d\big|_{X^n}$. Any such map is null-homotopic precisely when the corresponding elements in the kernel is a coboundary. So there is a natural isomorphism: $\mathbf{Hom}(S(n), X)/\simeq \; \cong H^n(X)$.

By using the free cdga functor we can turn these cochain complexes into cdga's $\Lambda D(n)$ and $\Lambda S(n)$. So $\Lambda D(n)$ consists of linear combinations of b^k and cb^k when n is even, and it consists of linear combinations of $c^k b$ and c^k when n is odd. In both cases we can compute the differentials using the Leibniz rule:

$$d(b^k) = k \cdot cb^{k-1} \qquad d(c^k b) = c^{k+1}$$
$$d(cb^k) = 0 \qquad d(c^k) = 0$$

Those cocycles are in fact coboundaries (using that $\mathbb{k}$ is a field of characteristic 0):

$$cb^k = \frac{1}{k} d(b^{k+1})$$
$$c^k = d(bc^{k-1})$$

There are no additional cocycles in $\Lambda D(n)$ besides the constants and c. So we conclude that $\Lambda D(n)$ is acyclic as an augmented algebra. In other words $\Lambda(j_n) : \mathbb{k} \to \Lambda D(n)$ is a quasi isomorphism.

The situation for $\Lambda S(n)$ is easier as it has only one generator (as algebra). For even n this means it is given by polynomials in a. For odd n it is an exterior algebra, meaning $a^2 = 0$. Again the sets $\Lambda(I) = \{\Lambda(i_n) : \Lambda S(n+1) \to \Lambda D(n) \mid n \in \mathbb{N}\}$ and $\Lambda(J) = \{\Lambda(j_n) : \mathbb{k} \to \Lambda D(n) \mid n \in \mathbb{N}\}$ play an important role.

Theorem 4.1.2. *The sets $\Lambda(I)$ and $\Lambda(J)$ generate a model structure on $\mathbf{CDGA}_{\mathbb{k}}$ where:*

- *weak equivalences are quasi isomorphisms,*
- *fibrations are (degree wise) surjective maps and*
- *cofibrations are maps with the left lifting property against trivial fibrations.*

We will prove this theorem in the next section. Note that the functors Λ and U thus form a Quillen pair with this model structure.

4.1.1 *Why we need $char(\mathbb{k}) = 0$ for algebras*

The above Quillen pair (Λ, U) fails to be a Quillen pair if $char(\mathbb{k}) = p \neq 0$. We will show this by proving that the maps $\Lambda(j_n)$ are not weak equivalences for even n. Consider $b^p \in \Lambda D(n)$, then by the Leibniz rule:

$$d(b^p) = p \cdot cb^{p-1} = 0.$$

So b^p is a cocycle. Now assume $b^p = dx$ for some x of degree $pn - 1$, then x contains a factor c for degree reasons. By the calculations above we see that any element containing c has a trivial differential or has a factor c in its differential, contradicting $b^p = dx$. So this cocycle is not a coboundary and $\Lambda D(n)$ is not acyclic.

4.2 the quillen model structure on **CDGA**

In this section we will define a model structure on cdga's over a field $\mathbb{k}$ of characteristic zero, where the weak equivalences are quasi isomorphisms and fibrations are surjective maps. The cofibrations are defined to be the maps with a left lifting property with respect to trivial fibrations.

Proposition 4.2.1. *There is a model structure on* $\mathbf{CDGA}_{\mathbb{k}}$ *where* $f : A \to B$ *is*

- *a* weak equivalence *if f is a quasi isomorphism,*
- *a* fibration *if f is an surjective and*
- *a* cofibration *if f has the LLP w.r.t. trivial fibrations*

We will prove the different axioms in the following lemmas. First observe that the classes as defined above are indeed closed under composition and contain all isomorphisms.

Note that with these classes, every cdga is a fibrant object.

Lemma 4.2.2 (MC1)**.** *The category has all finite limits and colimits.*

Proof. As discussed earlier products are given by direct sums and equalizers are kernels. Furthermore the coproducts are tensor products and coequalizers are quotients. □

Lemma 4.2.3 (MC2)**.** *The* 2-out-of-3 *property for quasi isomorphisms.*

Proof. Let f and g be two maps such that two out of f, g and fg are weak equivalences. This means that two out of $H(f)$, $H(g)$ and $H(f)H(g)$ are isomorphisms. The 2-out-of-3 property holds for isomorphisms, proving the statement. □

Lemma 4.2.4 (MC3)**.** *All three classes are closed under retracts*

Proof. For the class of weak equivalences and fibrations this follows easily from basic category theory. For cofibrations we consider the following diagram where the horizontal compositions are identities:

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "g"] & A \arrow[r] \arrow[d, hook, "f"] & A' \arrow[d, "g"] \\ B' \arrow[r] & B \arrow[r] & B' \end{tikzcd}$$

We need to prove that g is a cofibration, so for any lifting problem with a trivial fibration we need to find a lift. We are in the following situation:

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "g"] & A \arrow[r] \arrow[d, hook, "f"] & A' \arrow[r] \arrow[d, "g"] & X \arrow[d, two heads, "\sim"] \\ B' \arrow[r] & B \arrow[r] & B' \arrow[r] & Y \end{tikzcd}$$

Now we can find a lift starting at B, since f is a cofibration. By precomposition we obtain a lift $B' \to X$. □

Next we will prove the factorization property [MC5]. We will prove one part directly and the other by Quillen's small object argument. When proved, we get an easy way to prove the missing lifting property of [MC4]. For the Quillen's small object argument we use a class of generating cofibrations.

Definition 4.2.5. Define the following objects and sets of maps:

- $\Lambda S(n)$ is the cdga generated by one element a of degree n such that $da = 0$.
- $\Lambda D(n)$ is the cdga generated by two elements b and c of degree n and $n+1$ respectively, such that $db = c$ (and necessarily $dc = 0$).
- $I = \{i_n : \mathbb{k} \to \Lambda D(n) \mid n \in \mathbb{N}\}$ is the set of units.
- $J = \{j_n : \Lambda S(n+1) \to \Lambda D(n) \mid n \in \mathbb{N}\}$ is the set of inclusions j_n defined by $j_n(a) = b$.

Lemma 4.2.6. *[MC5a] A map $f : A \to X$ can be factorized as $f = pi$ where i is a trivial cofibration and p a fibration.*

Proof. Consider the free cdga $C = \bigotimes_{x \in X} T(|x|)$. There is an obvious surjective map $p : C \to X$ which sends a generator corresponding to x to x. Now define maps ϕ and ψ in

$$A \xrightarrow{\phi} A \otimes C \xrightarrow{\psi} X$$

by $\phi(a) = a \otimes 1$ and $\psi(a \otimes c) = f(a) \cdot p(c)$. Now ψ is clearly surjective (as p is) and ϕ is clearly a weak equivalence (by the Künneth theorem). Furthermore ϕ is a cofibration as we can construct lifts using the freeness of C. □

Remark 4.2.7. *The map ϕ in the above construction has a left inverse $\overline{\phi}$ given by $\overline{\phi}(x \otimes c) = x \cdot \epsilon(c)$, where ϵ is the natural augmentation of a free cdga (i.e. it send 1 to 1 and all generators to 0). Clearly $\overline{\phi}\phi = \mathbf{id}$, and so $\overline{\phi}$ is a fibration as well.*

Furthermore, if f is a weak equivalence then by the 2-out-of-3 property both ϕ and ψ are weak equivalences. Applying it once more, we find that $\overline{\phi}$ too is a weak equivalence. So for any weak equivalence $f : A \to X$ we find trivial fibrations $\overline{\phi} : A \otimes C \twoheadrightarrow A$ and $\psi : A \otimes C \twoheadrightarrow X$ compatible with f.

Lemma 4.2.8. *The maps i_n are trivial cofibrations and the maps j_n are cofibrations.*

Proof. Since $H(\Lambda D(n)) = \Bbbk$ (as stated earlier this uses $char(\Bbbk) = 0$) we see that indeed $H(i_n)$ is an isomorphism. For the lifting property of i_n and j_n simply use surjectivity of the fibrations and the freeness of $\Lambda D(n)$ and $\Lambda S(n)$. □

Lemma 4.2.9. *The class of cofibrations is saturated.*

Proof. We need to prove that the classes are closed under retracts (this is already done), pushouts and transfinite compositions. For the class of cofibrations, this is easy as they are defined by the LLP and colimits behave nice with respect to such classes. □

As a consequence of the above two lemmas, the class generated by J is contained in the class of cofibrations. We can characterize trivial fibrations with J.

Lemma 4.2.10. *If $p : X \to Y$ has the RLP w.r.t. J then p is a trivial fibration.*

Proof. Let $y \in Y$ be of degree n and dy its boundary. By assumption we can find a lift in the following diagram:

$$\begin{tikzcd} \Lambda S(n+1) \arrow[r, "a \mapsto 0"] \arrow[d, hook, "j_n"] & X \arrow[d, "f"] \\ \Lambda D(n) \arrow[r, "b \mapsto dy"] & Y \end{tikzcd}$$

The lift $h : D(n) \to X$ defines a preimage $x' = h(b)$ for dy. Now we can define a similar square to find a preimage x of y as follows:

$$\begin{tikzcd} \Lambda S(n) \arrow[r, "a \mapsto x'"] \arrow[d, hook, "j_{n-1}"] & X \arrow[d, "f"] \\ \Lambda D(n-1) \arrow[r, "b \mapsto y"] & Y \end{tikzcd}$$

The lift $h : D(n-1) \to X$ defines $x = h(b)$. This proves that f is surjective. Note that $dx = x'$.

Now if $[y] \in H(Y)$ is some class, then $dy = 0$, and so by the above we find a preimage x of y such that $dx = 0$, proving that $H(f)$ is surjective. Now let $[x] \in H(X)$ such that $[f(x)] = 0$, then there is an element β such that $f(x) = d\beta$, again by the above we can lift β to get $x = d\alpha.$, hence $H(f)$ is injective. Conclude that f is a trivial fibration. □

We can use Quillen's small object argument with the set J. The argument directly proves the following lemma. Together with the above lemmas this translates to the required factorization.

Lemma 4.2.11. *A map $f : A \to X$ can be factorized as $f = pi$ where i is in the class generated by J and p has the RLP w.r.t. J.*

Proof. This follows from Quillen's small object argument. □

Corollary 4.2.12. *[MC5b] A map $f : A \to X$ can be factorized as $f = pi$ where i is a cofibration and p a trivial fibration.*

Lemma 4.2.13. *[MC4] The lifting properties.*

Proof. One part is already established by definition (cofibrations are defined by an LLP). It remains to show that we can lift in the following situation:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\simeq f"] & X \arrow[d, two heads] \\ B \arrow[r] & Y \end{tikzcd}$$

Now factor $f = pi$, where p is a fibration and i a trivial cofibration. By the 2-out-of-3 property p is also a weak equivalence and we can find a lift in the following diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, hook, "f"] & Z \arrow[d, two heads, "\simeq p"] \\ B \arrow[r, "\mathbf{id}"] \arrow[ur, dashed] & B \end{tikzcd}$$

This defines f as a retract of i. Now we know that i has the LLP w.r.t. fibrations (by the small object argument above), hence f has the LLP w.r.t. fibrations as well. □

4.3 HOMOTOPY RELATIONS ON **CDGA**

Although the abstract theory of model categories gives us tools to construct a homotopy relation (Definition B.1.14), it is useful to have a concrete notion of homotopic maps.

Consider the free cdga on one generator $\Lambda(t, dt)$, where $|t| = 0$, this can be thought of as the (dual) unit interval with endpoints 1 and t. Notice that this cdga is isomorphic to $\Lambda(D(0))$ as defined in the previous section. We define two *endpoint maps* as follows:

$$d_0, d_1 : \Lambda(t, dt) \to \mathbb{k}$$

$$d_0(t) = 1, \qquad d_1(t) = 0,$$

this extends linearly and multiplicatively. Note that it follows that we have $d_0(1-t) = 0$ and $d_1(1-t) = 1$. These two functions extend to tensor products as $d_0, d_1 : \Lambda(t, dt) \otimes X \to \mathbb{k} \otimes X \xrightarrow{\cong} X$.

Definition 4.3.1. We call $f, g : A \to X$ homotopic ($f \simeq g$) if there is a map

$$h : A \to \Lambda(t, dt) \otimes X,$$

such that $d_0 h = g$ and $d_1 h = f$.

In terms of model categories, such a homotopy is a right homotopy and the object $\Lambda(t, dt) \otimes X$ is a path object for X. We can see as follows that it is a very good path object (Definition B.1.7). First note that $\Lambda(t, dt) \otimes X \xrightarrow{(d_0, d_1)} X \oplus X$ is surjective (for $(x, y) \in X \oplus X$ take $t \otimes x + (1-t) \otimes y$). Secondly we note that $\Lambda(t, dt) = \Lambda(D(0))$ and hence $\mathbb{k} \to \Lambda(t, dt)$ is a cofibration, by Lemma B.0.11 we have that $X \to \Lambda(t, dt) \otimes X$ is a (necessarily trivial) cofibration.

Clearly we have that $f \simeq g$ implies $f \simeq^r g$ (see Definition B.1.9), however the converse need not be true.

Lemma 4.3.2. *If A is a cofibrant cdga and $f \simeq^r g : A \to X$, then $f \simeq g$ in the above sense.*

Proof. Because A is cofibrant, there is a very good homotopy H. Consider a lifting problem to construct a map $Path_X \to \Lambda(t, dt) \otimes X$. □

Corollary 4.3.3. *For cofibrant A, $\simeq$ defines a equivalence relation.*

Definition 4.3.4. For cofibrant A define the set of equivalence classes as:

$$[A, X] = \mathbf{Hom}_{\mathbf{CDGA}_{\Bbbk}}(A, X)/ \simeq .$$

The results from model categories immediately imply the following results. Here we use Lemma B.1.6, B.1.12 and B.1.15.

Corollary 4.3.5. *Let A be cofibrant.*

- *Let $i : A \to B$ be a trivial cofibration, then the induced map $i^* : [B, X] \to [A, X]$ is a bijection.*
- *Let $p : X \to Y$ be a trivial fibration, then the induced map $p_* : [A, X] \to [A, Y]$ is a bijection.*
- *Let A and X both be cofibrant, then $f : A \xrightarrow{\simeq} X$ is a weak equivalence if and only if f is a strong homotopy equivalence. Moreover, the two induced maps are bijections:*

$$f_* : [Z, A] \xrightarrow{\cong} [Z, X],$$
$$f^* : [X, Z] \xrightarrow{\cong} [A, X].$$

Remark 4.3.6. *By Remark 4.2.7 we can generalize the second item to arbitrary weak equivalences: If A is cofibrant and $f : X \to Y$ a weak equivalence, then the induced map $f_* : [A, X] \to [A, Y]$ is a bijection, as seen from the following diagram:*

$$\begin{tikzcd} & {[A, X \otimes C]} \arrow[dl, "\overline{\phi}_*"', "\cong"] \arrow[dr, "\psi_*", "\cong"'] & \\ {[A, X]} \arrow[rr, "f_*"] & & {[A, Y]} \end{tikzcd}$$

Lemma 4.3.7. *Let $f, g : A \to X$ be two homotopic maps, then $H(f) = H(g) : HA \to HX$.*

Proof. Let h be the homotopy such that $f = d_1 h$ and $g = d_0 h$. By the Künneth theorem we get the following commuting square for $i = 0, 1$:

$$\begin{tikzcd} H(\Lambda(t, dt)) \otimes H(A) \arrow[r, "d_i \otimes \mathbf{id}"] \arrow[d, "\cong"] & \Bbbk \otimes H(A) \arrow[d, "\cong"] \\ H(\Lambda(t, dt) \otimes A) \arrow[r, "d_i"] & H(\Bbbk \otimes A) \end{tikzcd}$$

Now we know that $H(d_0) = H(d_1) : H(\Lambda(t, dt)) \to \Bbbk$ as $\Lambda(t, dt)$ is acyclic and the induced map sends 1 to 1. So the two bottom maps in the diagram are equal as well. Now we conclude $H(f) = H(d_1)H(h) = H(d_0)H(h) = H(g)$. $\square$

4.4 HOMOTOPY THEORY OF AUGMENTED CDGA'S

Recall that an augmented cdga is a cdga A with an algebra map $A \xrightarrow{\epsilon} \mathbb{k}$ (this implies that $\epsilon\eta = \mathbf{id}$). This is precisely the dual notion of a pointed space. We will use the general fact that if $\mathbf{C}$ is a model category, then the over (resp. under) category $\mathbf{C}/A$ (resp. $A/\mathbf{C}$) for any object A admit an induced model structure. In particular, the category of augmented cdga's (with augmentation preserving maps) has a model structure with the fibrations, cofibrations and weak equivalences as above.

Although the model structure is completely induced, it might still be fruitful to discuss the right notion of a homotopy for augmented cdga's. Consider the following pullback of cdga's:

$$\begin{tikzcd} \Lambda(t,dt)\overline{\otimes}A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda(t,dt)\otimes A \arrow[d, "\mathbf{id}\otimes\epsilon"] \\ \mathbb{k} \arrow[r] & \Lambda(t,dt)\otimes\mathbb{k} \end{tikzcd}$$

The pullback is the subspace of elements $x \otimes a$ in $\Lambda(t,dt)\otimes A$ such that $x \cdot \epsilon(a) \in \mathbb{k}$. Note that this construction is dual to a construction on topological spaces: in order to define a homotopy which is constant on the point x_0, we define the homotopy to be a map from a quotient $X \times I / x_0 \times I$.

Definition 4.4.1. Two maps $f, g : A \to X$ between augmented cdga's are said to be *homotopic* if there is a map

$$h : A \to \Lambda(t,dt)\overline{\otimes}X$$

such that $d_0 h = g$ and $d_1 h = f$.

In the next section homotopy groups of augmented cdga's will be defined. In order to define this we first need another tool.

Definition 4.4.2. Define the *augmentation ideal* of A as $\overline{A} = \ker\epsilon$. Define the *cochain complex of indecomposables* of A as $QA = \overline{A}/\overline{A}\cdot\overline{A}$.

The first observation one should make is that Q is a functor from algebras to modules (or differential algebras to differential modules) which is particularly nice for free (differential) algebras, as we have that $Q\Lambda V = V$ for any (differential) module V.

The second observation is that Q is nicely behaved on tensor products and cokernels.

Lemma 4.4.3. *Let A and B be two augmented cdga's, then there is a natural isomorphism*

$$Q(A \otimes B) \cong Q(A) \oplus Q(B).$$

Proof. First note that the augmentation ideal is expressed as $\overline{A \otimes B} = \overline{A} \otimes B + A \otimes \overline{B}$ and the product is $\overline{A \otimes B} \cdot \overline{A \otimes B} = \overline{A} \otimes \overline{B} + \overline{A} \cdot \overline{A} \otimes \Bbbk + \Bbbk \otimes \overline{B}$. With this we can prove the statement

$$\begin{aligned} Q(A \otimes B) &= \frac{\overline{A} \otimes B + A \otimes \overline{B}}{\overline{A} \otimes \overline{B} + \overline{A} \cdot \overline{A} \otimes \Bbbk + \Bbbk \otimes \overline{B}} \\ &\cong \frac{\overline{A} \otimes \Bbbk \oplus \Bbbk \otimes \overline{B}}{\overline{A} \cdot \overline{A} \otimes \Bbbk \oplus \Bbbk \otimes \overline{B} \cdot \overline{B}} \cong Q(A) \oplus Q(B). \end{aligned}$$

□

Lemma 4.4.4. *Let $f : A \to B$ be a map of augmented cdga's, then there is a natural isomorphism*

$$Q(\mathrm{coker}(f)) \cong \mathrm{coker}(Qf).$$

Proof. First note that the cokernel of f in the category of augmented cdga's is $\mathrm{coker}(f) = B/f(\overline{A})B$ and that its augmentation ideal is $\overline{B}/f(\overline{A})B$, where $f(\overline{A})B$ is the ideal generated by $f(\overline{A})$. Just as above we make a simple calculation, where $p : \overline{B} \to Q(B)$ is the projection map:

$$\begin{aligned} Q(\mathrm{coker}(f)) &= \frac{\overline{B}/f(\overline{A})B}{\overline{B}/f(\overline{A})B \cdot \overline{B}/f(\overline{A})B} \\ &\cong \frac{\overline{B}/\overline{B} \cdot \overline{B}}{pf(\overline{A})B} = \frac{Q(B)}{Qf(Q(A))}. \end{aligned}$$

□

Corollary 4.4.5. *Combining the two lemmas above, we see that Q (as functor from augmented cdga's to cochain complexes) preserves pushouts.*

Furthermore we have the following lemma which is of homotopical interest.

Lemma 4.4.6. *If $f : A \to B$ is a cofibration of augmented cdga's, then Qf is injective in positive degrees.*

Proof. First we define an augmented cdga $U(n)$ for each positive n as $U(n) = D(n) \oplus \Bbbk$ with trivial multiplication and where the term $\Bbbk$ is used for the unit and augmentation. Notice that the map $U(n) \to \Bbbk$ is a trivial fibration. By the lifting property we see that the induced map

$$\mathbf{Hom}_{\mathbf{CDGA}^*}(B, U(n)) \xrightarrow{f^*} \mathbf{Hom}_{\mathbf{CDGA}^*}(A, U(n))$$

is surjective for each positive n. Note that maps from A to $U(n)$ will send products to zero and that it is fixed on the augmentation. So there is a natural isomorphism $\mathbf{Hom}_{\mathbf{CDGA}^*}(A, U(n)) \cong \mathbf{Hom}_{\Bbbk}(Q(A)^n, \Bbbk)$. Hence

$$\mathbf{Hom}_{\Bbbk}(Q(B)^n, \Bbbk) \xrightarrow{(Qf)^*} \mathbf{Hom}_{\Bbbk}(Q(A)^n, \Bbbk)$$

is surjective, and so Qf itself is injective in positive n. □

4.5 HOMOTOPY GROUPS OF CDGA'S

As the eventual goal is to compare the homotopy theory of spaces with the homotopy theory of cdga's, it is natural to investigate an analogue of homotopy groups in the category of cdga's. In topology we can only define homotopy groups on pointed spaces, dually we will consider augmented cdga's in this section.

Definition 4.5.1. The *homotopy groups of an augmented cdga A* are

$$\pi^i(A) = H^i(QA).$$

This construction is functorial (since both Q and H are) and, as the following lemma shows, homotopy invariant.

Lemma 4.5.2. *Let $f : A \to X$ and $g : A \to X$ be a maps of augmented cdga's. If f and g are homotopic, then the induced maps are equal:*

$$f_* = g_* : \pi_*(A) \to \pi_*(X).$$

Proof. Let $h : A \to \Lambda(t, dt) \otimes X$ be a homotopy. We will, just as in Lemma 4.3.7, prove that the maps $HQ(d_0)$ and $HQ(d_1)$ are equal, then it follows that $HQ(f) = HQ(d_1 h) = HQ(d_0 h) = HQ(g)$.

Using Lemma 4.4.3 we can identify the induced maps $Q(d_i) : Q(\Lambda(t, dt) \otimes X) \to Q(X)$ with maps

$$Q(d_i) : Q(\Lambda(t, dt)) \oplus Q(A) \to Q(A).$$

Now $Q(\Lambda(t, dt)) = D(0)$ and hence it is acyclic, so when we pass to homology, this term vanishes. In other words both maps $d_{i*} : H(D(0)) \oplus H(Q(A)) \to H(Q(A))$ are the identity maps on $H(Q(A))$. $\square$

Consider the augmented cdga $V(n) = S(n) \oplus \mathbb{k}$, with trivial multiplication and where the term $\mathbb{k}$ is used for the unit and augmentation. This augmented cdga can be thought of as a specific model of the sphere. In particular the homotopy groups can be expressed as follows.

Lemma 4.5.3. *There is a natural bijection for any augmented cdga A*

$$[A, V(n)] \xrightarrow{\cong} \mathbf{Hom}_{\mathbb{k}}(\pi^n(A), \mathbb{k}).$$

Proof. Note that $Q(V(n))$ in degree n is just $\mathbb{k}$ and 0 in the other degrees, so its homotopy groups consists of a single $\mathbb{k}$ in degree n. This establishes the map:

$$\pi^n : \mathbf{Hom}(A, V(n)) \to \mathbf{Hom}_{\mathbb{k}}(\pi^n(A), \mathbb{k}).$$

Now by Lemma 4.5.2 we get a map from the set of homotopy classes $[A, V(n)]$ instead of the **Hom**-set. It remains to prove that the map is an isomorphism. Surjectivity follows easily. Given a map $f : \pi^n(A) \to \mathbb{k}$, we can extend this to $A \to V(n)$ because the multiplication on $V(n)$ is trivial.

For injectivity suppose $\phi, \psi : A \to V(n)$ be two maps such that $\pi^n(\phi) = \pi^n(\psi)$. We will first define a chain homotopy $D : A^* \to V(n)^{*-1}$, for this we only need to specify the map $D^n : A^{n+1} \to V(n)^n = \mathbb{Q}$. Decompose the vector space A^{n+1} as $A^{n+1} = \operatorname{im} d \oplus V$ for some V. Now set $D^n(v) = 0$ for all $v \in V$ and $D^n(db) = \phi(b) - \psi(b)$. We should check that D is well defined. Note that for cycles we get $\phi(c) = \psi(c)$, as $H(Q(\phi)) = H(Q(\psi))$. So if $db = dc$, then we get $D(db) = \phi(b) - \psi(b) = \phi(c) - \psi(c) = D(dc)$, i.e. D is well defined. We can now define a map of augmented cdga's:

$$\begin{aligned} h : X &\to \Lambda(t, dt) \overline{\otimes} V(n) \\ x &\mapsto dt \otimes D(x) + 1 \otimes \phi(x) - t \otimes \phi(x) + t \otimes \psi(x) \end{aligned}$$

This map commutes with the differential by the definition of D. Now we see that $d_0 h = \psi$ and $d_1 h = \phi$. Hence the two maps represent the same class, and we have proven the injectivity. $\square$

From now on the dual of a vector space will be denoted as $V^* = \mathbf{Hom}_{\mathbb{k}}(V, \mathbb{k})$. So the above lemma states that there is a bijection $[A, V(n)] \cong \pi^n(A)^*$.

In topology we know that a fibration induces a long exact sequence of homotopy groups. In the case of cdga's a similar long exact sequence for a cofibration will exist.

Lemma 4.5.4. *Given a pushout square of augmented cdga's*

$$\begin{tikzcd} A \arrow[r, hook, "g"] \arrow[d, "f"] & C \arrow[d, "i"] \\ B \arrow[r, "j"] \arrow[ur, phantom, "\ulcorner"] & P \end{tikzcd}$$

where g is a cofibration. There is a natural long exact sequence

$$\pi^o(V) \xrightarrow{(f_*,g_*)} \pi^0(B) \oplus \pi^0(C) \xrightarrow{j_*-i_*} \pi^0(P) \xrightarrow{\partial} \pi^1(A) \to \cdots$$

Proof. First note that j is also a cofibration. By Lemma 4.4.6 the maps Qg and Qj are injective in positive degrees. By applying Q we get two exact sequence (in positive degrees) as shown in the following diagram. By the fact that Q preserves pushouts (Corollary 4.4.5) the cokernels coincide.

$$\begin{tikzcd} 0 \arrow[r] & Q(A) \arrow[r] \arrow[d] & Q(C) \arrow[r] \arrow[d] & \operatorname{coker}(f_*) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Q(B) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Q(P) \arrow[r] & \operatorname{coker}(f_*) \arrow[r] & 0 \end{tikzcd}$$

Now the well known Mayer-Vietoris exact sequence can be constructed. This proves the statement. $\square$

Corollary 4.5.5. *When we take $B = \mathbb{k}$ in the above situation, we get a long exact sequence*

$$\pi^0(A) \xrightarrow{g_*} \pi^0(C) \to \pi^0(\operatorname{coker}(g)) \to \pi^1(A) \to \cdots$$

5 POLYNOMIAL FORMS

5.1 CDGA OF POLYNOMIALS

We will now give a cdga model for the n-simplex Δ^n. This then allows for simplicial methods. In the following definition one should remember the topological n-simplex defined as convex span.

Definition 5.1.1. For all $n \in \mathbb{N}$ define the following cdga:

$$(A_{PL})_n = \frac{\Lambda(x_0, \ldots, x_n, dx_0, \ldots, dx_n)}{(\sum_{i=0}^n x_i - 1, \sum_{i=0}^n dx_i)},$$

where $|x_i| = 0$. So it is the free cdga with $n+1$ generators and their differentials such that $\sum_{i=0}^n x_i = 1$ and in order to be well behaved $\sum_{i=0}^n dx_i = 0$.

Note that the inclusion $\Lambda(x_1, \ldots, x_n, dx_1, \ldots, dx_n) \to A_{PLn}$ is an isomorphism of cdga's. So A_{PLn} is free and (algebra) maps from it are determined by their images on x_i for $i = 1, \ldots, n$ (also note that this determines the images for dx_i). This fact will be used throughout. Also note that we have already seen the dual unit interval $\Lambda(t, dt)$ which is isomorphic to A_{PL1}.

These cdga's will assemble into a simplicial cdga when we define the face and degeneracy maps as follows ($j = 1, \ldots, n$):

$$d_i(x_j) = \begin{cases} x_{j-1}, & \text{if } i < j \\ 0, & \text{if } i = j \\ x_j, & \text{if } i > j \end{cases} \qquad d_i : A_{PLn} \to A_{PLn-1}$$

$$s_i(x_j) = \begin{cases} x_{j+1}, & \text{if } i < j \\ x_j + x_{j+1}, & \text{if } i = j \\ x_j, & \text{if } i > j \end{cases} \qquad s_i : A_{PLn} \to A_{PLn+1}$$

One can check that $A_{PL} \in \mathbf{sCDGA}_{\mathbb{k}}$. We will denote the subspace of homogeneous elements of degree k as ${A_{PL}}^k$, this is a simplicial $\mathbb{k}$-module as the maps d_i and s_i are graded maps of degree 0.

Lemma 5.1.2. *${A_{PL}}^k$ is contractible.*

Proof. We will prove this by defining an extra degeneracy $s : A_{PLn} \to A_{PLn+1}$. In the more geometric context of topological n-simplices we would achieve this by dividing by $1 - x_0$. However, since this algebra consists of polynomials only, this cannot be done. Instead, we will multiply everything by $(1 - x_0)^2$, so that we can divide by $1 - x_0$. Define for $i = 1, \dots, n$:

$$s(1) = (1 - x_0)^2$$
$$s(x_i) = (1 - x_0) \cdot x_{i+1}$$

Extend on the differentials and multiplicatively on A_{PLn}. As $s(1) \neq 1$ this map is not an algebra map, however it well-defined as a map of cochain complexes. In particular when restricted to degree k we get a linear map:

$$s : A_{PL}{}_n^k \to A_{PL}{}_{n+1}^k.$$

Proving the necessary properties of an extra degeneracy is fairly easy. For $n \geq 1$ we get (on generators):

$$\begin{aligned} d_0 s(1) &= d_0(1 - x_0)^2 = (1 - 0) \cdot (1 - 0) = 1 \\ d_0 s(x_i) &= d_0((1 - x_0)x_{i+1}) = d_0(1 - x_0) \cdot x_i \\ &= (1 - 0) \cdot x_i = x_i \end{aligned}$$

So $d_0 s = \mathbf{id}$.

$$\begin{aligned} d_{i+1}s(1) &= d_{i+1}(1 - x_0)^2 = d_{i+1}(\sum_{j=1}^{n} x_j)^2 \\ &= (\sum_{j=1}^{n-1} x_j)^2 = (1 - x_0)^2 = sd_i(1) \\ d_{i+1}s(x_j) &= d_{i+1}(1 - x_0)d_{i+1}(x_j) = (1 - x_0)d_i(x_{j+1}) = sd_i(x_j) \end{aligned}$$

So $d_{i+1}s = sd_i$. Similarly $s_{i+1}s = ss_i$. And finally for $n = 0$ we have $d_1 s = 0$.

So we have an extra degeneracy $s : {A_{PL}}^k \to {A_{PL}}^k$, and hence (see for example [GJ99]) we have that ${A_{PL}}^k$ is contractible. As a consequence ${A_{PL}}^k \to *$ is a weak equivalence. □

Lemma 5.1.3. *${A_{PL}}^k$ is a Kan complex.*

Proof. By the simple fact that ${A_{PL}}^k$ is a simplicial group, it is a Kan complex [GJ99]. □

Combining these two lemmas gives us the following.

Corollary 5.1.4. *${A_{PL}}^k \to *$ is a trivial fibration in the standard model structure on* **sSet**.

Besides the simplicial structure of A_{PL}, there is also the structure of a cochain complex.

Lemma 5.1.5. *A_{PLn} is acyclic, i.e. $H(A_{PLn}) = \mathbb{k} \cdot [1]$.*

Proof. This is clear for $A_{PL0} = \mathbb{k} \cdot 1$. For A_{PL1} we see that $A_{PL1} = \Lambda(x_1, dx_1) \cong \Lambda D(0)$, which we proved to be acyclic in the previous section.

For general n we can identify $A_{PLn} \cong \bigotimes_{i=1}^n \Lambda(x_i, dx_i)$, because Λ is left adjoint and hence preserves coproducts. By the Künneth theorem Theorem 1.1.5 we conclude $H(A_{PLn}) \cong \bigotimes_{i=1}^n H\Lambda(x_i, dx_i) \cong \bigotimes_{i=1}^n H\Lambda D(0) \cong \mathbb{k} \cdot [1]$.

So indeed A_{PLn} is acyclic for all n. □

5.2 POLYNOMIAL FORMS ON A SPACE

There is a general way to construct contravariant functors from **sSet** whenever we have some simplicial object. In our case we have the simplicial cdga A_{PL} (which is nothing more than a functor $\mathbf{\Delta}^{\text{op}} \to \mathbf{CDGA}$) and we want to extend to a contravariant functor $\mathbf{sSet} \to \mathbf{CDGA}_{\mathbb{k}}$. This will be done via *Kan extensions*.

Given a category $\mathbf{C}$ and a functor $F : \mathbf{\Delta} \to \mathbf{C}$, then define the following on objects:

$$F_!(X) = \operatorname*{colim}_{\Delta[n] \to X} F[n] \qquad\qquad X \in \mathbf{sSet}$$

$$F^*(C)_n = \mathbf{Hom}_{\mathbf{C}}(F[n], Y) \qquad\qquad C \in \mathbf{C}$$

A simplicial map $X \to Y$ induces a map of the diagrams of which we take colimits. Applying F on these diagrams, make it clear that $F_!$ is functorial. Secondly we see readily that F^* is functorial. By using the definition of colimit and the Yoneda lemma (Y) we can prove that $F_!$ is left adjoint to F^* by the following calculation:

$$\begin{aligned}
\mathbf{Hom}_{\mathbf{C}}(F_!(X), Y) &\cong \mathbf{Hom}_{\mathbf{C}}(\operatorname*{colim}_{\Delta[n] \to X} F[n], Y) \\
&\cong \lim_{\Delta[n] \to X} \mathbf{Hom}_{\mathbf{C}}(F[n], Y) \\
&\cong \lim_{\Delta[n] \to X} F^*(Y)_n
\end{aligned}$$

$$
\begin{aligned}
&\overset{Y}{\cong} \lim_{\Delta[n]\to X} \mathbf{Hom}_{\mathbf{sSet}}(\Delta[n], F^*(Y)) \\
&\cong \mathbf{Hom}_{\mathbf{sSet}}(\operatorname*{colim}_{\Delta[n]\to X} \Delta[n], F^*(Y)) \\
&\cong \mathbf{Hom}_{\mathbf{sSet}}(X, F^*(Y))
\end{aligned}
$$

Furthermore we have $F_! \circ \Delta[-] \cong F$. In short we have the following:

$$
\begin{tikzcd}
\mathbf{\Delta} \arrow[d, "{\Delta[-]}"] \arrow[drr, "F"] & & \\
\mathbf{sSet} \arrow[rr, dashed, shift left, "F_!"] \arrow[rr, phantom, "\perp"] & & \mathbf{C} \arrow[ll, dashed, shift left, "F^*"]
\end{tikzcd}
$$

5.2.1 *The cochain complex of polynomial forms*

In our case where $F = A_{PL}$ and $\mathbf{C} = \mathbf{CDGA}_{\Bbbk}$ we get:

$$
\begin{tikzcd}
\mathbf{\Delta} \arrow[d, "{\Delta[-]}"] \arrow[drr, "A_{PL}"] & & \\
\mathbf{sSet} \arrow[rr, dashed, shift left, "A"] \arrow[rr, phantom, "\perp"] & & \mathbf{CDGA}_{\Bbbk}^{op} \arrow[ll, dashed, shift left, "K"]
\end{tikzcd}
$$

Note that we have the opposite category of cdga's, so the definition of A is in terms of a limit instead of colimit. This allows us to give a nicer description:

$$
\begin{aligned}
A(X) = \lim_{\Delta[n]\to X} A_{PLn} &\overset{Y}{\cong} \lim_{\Delta[n]\to X} \mathbf{Hom}_{\mathbf{sSet}}(\Delta[n], A_{PL}) \\
&\cong \mathbf{Hom}_{\mathbf{sSet}}(\operatorname*{colim}_{\Delta[n]\to X} \Delta[n], A_{PL}) = \mathbf{Hom}_{\mathbf{sSet}}(X, A_{PL}),
\end{aligned}
$$

where the addition, multiplication and differential are defined pointwise. Conclude that we have the following contravariant functors (which form an adjoint pair):

$$
\begin{aligned}
A(X) &= \mathbf{Hom}_{\mathbf{sSet}}(X, A_{PL}) & X &\in \mathbf{sSet} \\
K(C)_n &= \mathbf{Hom}_{\mathbf{CDGA}_{\Bbbk}}(C, A_{PLn}) & C &\in \mathbf{CDGA}_{\Bbbk}
\end{aligned}
$$

5.2.2 *The singular cochain complex*

Another way to model the n-simplex is by the singular cochain complex associated to the topological n-simplices. Define the following (non-commutative) dga's:

$$C_n = C^*(\Delta^n; \Bbbk),$$

where $C^*(\Delta^n; \Bbbk)$ is the (normalized) singular cochain complex of Δ^n with coefficients in $\Bbbk$. The inclusion maps $d^i : \Delta^n \to \Delta^{n+1}$ and the maps $s^i : \Delta^n \to \Delta^{n-1}$ induce face and degeneracy maps on the dga's C_n, turning C into a simplicial dga. Again we can extend this to functors by Kan extensions

$$\begin{tikzcd} \mathbf{\Delta} \arrow[d, "\Delta[-]"'] \arrow[dr, "C"] & \\ \mathbf{sSet} \arrow[r, dashed, shift left, "C^*"] \arrow[r, phantom, "\perp"] & \mathbf{DGA}_{\Bbbk}^{op} \arrow[l, dashed, shift left] \end{tikzcd}$$

This left adjoint functor $C^* : \mathbf{sSet} \to \mathbf{DGA_{\Bbbk}}^{\mathrm{op}}$ is (just as above) defined as $C^*(X) = \mathbf{Hom_{sSet}}(X, C^*(\Delta[-]; \Bbbk))$. To see that this is precisely the classical definition of the singular cochain complex, we make the following calculation.

$$\begin{aligned} C^*(X) &= \mathbf{Hom}(X, C^*(\Delta[n])) \\ &= \mathbf{Hom}(X, \mathbf{Hom}(N\mathbb{Z}\Delta[n], \Bbbk)) \\ &\overset{(1)}{\cong} \mathbf{Hom}(X, \Gamma(\Bbbk)) \\ &\cong \mathbf{Hom}(N\mathbb{Z}(X), \Bbbk) \end{aligned}$$

where $\mathbb{Z}$ is the free simplicial abelian group, N is the normalized chain complex (this is the Dold-Kan equivalence) and Γ its right adjoint. At (1) we use the definition of this right adjoint $\Gamma(C) = \mathbf{Hom}(N\mathbb{Z}\Delta[n], C)$. Now the conclusion of this calculation is that $C^*(X)$ is precisely the dual complex of $N\mathbb{Z}(X)$, which is the singular (normalized) chain complex.

We will relate A_{PL} and C in order to obtain a natural quasi isomorphism $A(X) \xrightarrow{\simeq} C^*(X)$ for every $X \in \mathbf{sSet}$. Furthermore this map preserves multiplication on the homology algebras.

5.2.3 *Integration and Stokes' theorem for polynomial forms*

In this section we will prove that the singular cochain complex is quasi isomorphic to the cochain complex of polynomial forms. In order to do so we will define an integration map $\int_n : {A_{PL}}_n^n \to \Bbbk$, which will induce a map $\oint_n : A_{PLn} \to C_n$. For the simplices $\Delta[n]$ we already showed the cohomology agrees by the acyclicity of $A_{PLn} = A(\Delta[n])$ (Lemma 5.1.5).

For any $v \in A_{PL}{}_n^n$, we can write v as $v = p(x_1, \ldots, x_n)dx_1 \ldots dx_n$ where p is a polynomial in n variables. If $\mathbb{Q} \subset \Bbbk \subset \mathbb{C}$ we can integrate geometrically on the n-simplex:

$$\int_n v = \int_0^1 \int_0^{1-x_n} \cdots \int_0^{1-x_2-\cdots-x_n} p(x_1, \ldots, x_n) dx_1 dx_2 \ldots dx_n,$$

which defines a well-defined linear map $\int_n : A_{PL}{}_n^n \to \Bbbk$. For general fields of characteristic zero we can define it formally on the generators of $A_{PL}{}_n^n$ (as vector space):

$$\int_n x_1^{k_1} \ldots x_n^{k_n} dx_1 \ldots dx_n = \frac{k_1! \ldots k_n!}{(k_1 + \cdots + k_n + n)!}.$$

Let x be a k-simplex of $\Delta[n]$. Then x induces a linear map $x^* : A_{PL n} \to A_{PL k}$. Now if we have an element $v \in A_{PL}{}_n^k$, then observe that $x^*(v) \in A_{PL}{}_k^k$ is an element we can integrate. Now define

$$\oint_n (v)(x) = (-1)^{\frac{k(k-1)}{2}} \int_n x^*(v).$$

Note that $\oint_n (v) : \Delta[n] \to \Bbbk$ is just a map of sets, so we can extend this linearly to chains on $\Delta[n]$ to obtain a linear map $\oint_n (v) : \mathbb{Z}\Delta[n] \to \Bbbk$. If x is a degenerate simplex $x = s_j x'$, then $x^*(v) = s_j^*(x'^*(v))$. Now $x'^*(v) \in A_{PL}{}_{k-1}^k = 0$ and so the integral vanishes on degenerate simplices. In other words we get $\oint_n (v) \in C_n$. By linearity of $\int_n$ and x^*, we have a linear map $\oint_n : A_{PL n} \to C_n$.

Next we will show that $\oint = \{\oint_n\}_n$ is a simplicial map and that each $\oint_n$ is a chain map, in other words $\oint : A_{PL} \to C$ is a simplicial chain map (of complexes). Let $\sigma : \Delta[n] \to \Delta[k]$, and $\sigma^* : A_{PL k} \to A_{PL n}$ its induced map. We need to prove $\oint_n \circ \sigma^* = \sigma^* \circ \oint_k$. We show this as follows:

$$\begin{aligned}
\oint_n (\sigma^* v)(x) &= (-1)^{\frac{l(l-1)}{2}} \int_l x^*(\sigma^*(v)) \\
&= (-1)^{\frac{l(l-1)}{2}} \int_l (\sigma \circ x)^*(v) \\
&= \oint_k (v)(\sigma \circ x) \\
&= (\oint_k (v) \circ \sigma)(x) = \sigma^*(\oint_k (v)(x))
\end{aligned}$$

For it to be a chain map, we need to prove $d \circ \oint_n = \oint_n \circ d$. This is precisely *Stokes' theorem* and any prove will apply here as well [BG76].

We now proved that $\oint$ is indeed a simplicial chain map. Note that $\oint_n$ need not to preserve multiplication, so it fails to be a map of cochain algebras. However $\oint(1) = 1$ and so the induced map on homology sends the class of 1 in $H(A_{PLn}) = \Bbbk \cdot [1]$ to the class of 1 in $H(C_n) = \Bbbk \cdot [1]$. We have proven the following lemma.

Lemma 5.2.1. *The map $\oint_n : A_{PLn} \to C_n$ is a quasi isomorphism for all n.*

Recall that we can identify A_{PLn} with $A(\Delta[n])$ and similarly for the singular cochain complex.

Corollary 5.2.2. *The induced map $\oint : A(\Delta[n]) \to C^*(\Delta[n])$ is a quasi isomorphism for all n.*

We will now prove that the map $\oint : A(X) \to C^*(X)$ is a quasi isomorphism for any space X. We will do this in several steps, the base case of simplices is already proven. With induction we will prove it for spaces with finitely many simplices. At last we will use a limit argument for the general case.

Theorem 5.2.3. *The induced map $\oint : A(X) \to C^*(X)$ is a natural quasi isomorphism.*

Proof. Assume we have a simplicial set X such that $\oint : A(X) \to C^*(X)$ is a quasi isomorphism. We can add a simplex by considering pushouts of the following form:

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & X'
\end{tikzcd}$$

We can apply our two functors to it, and use the natural transformation $\oint$ to obtain the following cube:

$$\begin{tikzcd}
A(X') \arrow[rr] \arrow[dr] \arrow[dd] & & A(\Delta[n]) \arrow[dr, "\simeq"] \arrow[dd, two heads] & \\
& C^*(X') \arrow[rr] \arrow[dd] & & C^*(\Delta[n]) \arrow[dd, two heads] \\
A(X) \arrow[rr] \arrow[dr, "\simeq"] & & A(\partial\Delta[n]) \arrow[dr, "\simeq"] & \\
& C^*(X) \arrow[rr] & & C^*(\partial\Delta[n])
\end{tikzcd}$$

Note that $A(\Delta[n]) \xrightarrow{\simeq} C^*(\Delta[n])$ by Corollary 5.2.2, $A(X) \xrightarrow{\simeq} C^*(X)$ by assumption and $A(\partial\Delta[n]) \xrightarrow{\simeq} C^*(\partial\Delta[n])$ by induction. Secondly note that both A and C^* send injective maps to surjective maps, so we get fibrations on the right side of the diagram. Finally note that the front square and back square are pullbacks, by adjointness of A and C^*. Apply the cube lemma (Lemma B.4.2) to conclude that also $A(X') \xrightarrow{\simeq} C^*(X')$.

This proves $A(X) \xrightarrow{\simeq} C^*(X)$ for any simplicial set with finitely many non-degenerate simplices. We can extend this to simplicial sets of finite dimension by attaching many simplices at once. For this we observe that both A and C^* send coproducts to products and that cohomology commutes with products:

$$H(A(\coprod_\alpha X_\alpha)) \cong H(\prod_\alpha A(X_\alpha)) \cong \prod_\alpha H(A(X_\alpha)),$$

$$H(C^*(\coprod_\alpha X_\alpha)) \cong H(\prod_\alpha C^*(X_\alpha)) \cong \prod_\alpha H(C^*(X_\alpha)).$$

This means that we can extend the previous argument to pushout of this form:

$$\begin{tikzcd} \coprod_{\alpha \in A} \partial\Delta[n] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", near end] & X \arrow[d] \\ \coprod_{\alpha \in A} \Delta[n] \arrow[r] & X' \end{tikzcd}$$

Finally we can write any simplicial set X as a colimit of finite dimensional ones as:

$$sk_0 X \hookrightarrow sk_1 X \hookrightarrow sk_2 \hookrightarrow \ldots \operatorname{colim} sk_n X = X,$$

where $sk_i X$ has no non-degenerate simplices in dimensions $n > i$. By the above $\oint$ gives a quasi isomorphism on all the terms $sk_i X$. So we are in the following situation:

$$\begin{tikzcd} A(X) = \lim_i A(sk_i X) \arrow[r, dashed, two heads] \arrow[d, "\oint"] & A(sk_2 X) \arrow[r, two heads] \arrow[d, "\simeq", "\oint"'] & A(sk_1 X) \arrow[r, two heads] \arrow[d, "\simeq", "\oint"'] & A(sk_0 X) \arrow[d, "\simeq", "\oint"'] \\ C^*(X) = \lim_i C^*(sk_i X) \arrow[r, dashed, two heads] & C^*(sk_2 X) \arrow[r, two heads] & C^*(sk_1 X) \arrow[r, two heads] & C^*(sk_0 X) \end{tikzcd}$$

We will define long exact sequences for both sequences in the following way. As the following construction is quite general, consider arbitrary cochain algebras B_i as follows:

$$B = \lim_i B_i \dashrightarrow\!\!\!\!\rightarrow B_2 \xrightarrow{b_1} B_1 \xrightarrow{b_0} B_0$$

Define a map $t : \prod_i B_i \to \prod_i B_i$ defined by $t(x_0, x_1, \ldots) = (x_0 + b_0(x_1), x_1 + b_1(x_2), \ldots)$. Note that t is surjective and that

$B \cong \ker(t)$. So we get the following natural short exact sequence and its associated natural long exact sequence in homology:

$$0 \to B \xrightarrow{i} \prod_i B_i \xrightarrow{t} \prod_i B_i \to 0,$$

$$\cdots \xrightarrow{\Delta} H^n(B) \xrightarrow{i_*} H^n(\prod_i B_i) \xrightarrow{t_*} H^n(\prod_i B_i) \xrightarrow{\Delta} \cdots$$

In our case we get two such long exact sequences with $\oint$ connecting them. As cohomology commutes with products we get isomorphisms on the left and right in the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(\prod_i A(sk_i X)) \arrow[r] \arrow[d, "\cong", "\oint"'] & H^n(A(X)) \arrow[r] \arrow[d, "\oint"] & H^n(\prod_i A(sk_i X)) \arrow[r] \arrow[d, "\cong", "\oint"'] & \cdots \\
\cdots \arrow[r] & H^{n-1}(\prod_i C^*(sk_i X)) \arrow[r] & H^n(C^*(X)) \arrow[r] & H^n(\prod_i C^*(sk_i X)) \arrow[r] & \cdots
\end{tikzcd}$$

So by the five lemma we can conclude that the middle morphism is an isomorphism as well, proving the isomorphism $H^n(A(X)) \xrightarrow{\cong} H^n(C^*(X))$ for all n. This proves the statement for all X. $\square$

6 MINIMAL MODELS

In this section we will discuss the so called minimal models. These cdga's enjoy the property that we can easily prove properties inductively. Moreover it will turn out that weakly equivalent minimal models are actually isomorphic.

Definition 6.0.4. A cdga (A, d) is a *Sullivan algebra* if

- $A = \Lambda V$ is free as a commutative graded algebra, and
- V has a filtration
$$0 = V(-1) \subset V(0) \subset V(1) \subset \cdots \subset \bigcup_{k \in \mathbb{N}} V(k) = V,$$
such that $d(V(k)) \subset \Lambda V(k-1)$.

A cdga (A, d) is a *minimal Sullivan algebra* if in addition

- d is decomposable, i.e. $\operatorname{im}(d) \subset \Lambda^{\geq 2} V$.

Definition 6.0.5. Let (A, d) be any cdga. A *(minimal) Sullivan model* is a (minimal) Sullivan algebra (M, d) with a weak equivalence:
$$(M, d) \xrightarrow{\simeq} (A, d).$$

We will often say *minimal model* or *minimal algebra* to mean minimal Sullivan model or minimal Sullivan algebra. Note that a minimal algebra is naturally augmented as it is free as an algebra. This will be used implicitly. In many cases we can take the degree of the elements in V to induce the filtration, as seen in the following lemma.

Lemma 6.0.6. *Let (A, d) be a cdga which is* 1*-reduced, such that $A = \Lambda V$ is free as cga. Then the differential d is decomposable if and only if (A, d) is a Sullivan algebra filtered by degree.*

Proof. Let V be filtered by degree: $V(k) = V^{\leq k}$. Now $d(v) \in \Lambda V^{<k}$ for any $v \in V^k$. For degree reasons $d(v)$ is a product, so d is decomposable.

For the converse take $V(n) = V^{\leq n}$ (note that $V^0 = V^1 = 0$). Since d is decomposable we see that for $v \in V^n$: $d(v) = x \cdot y$ for some $x, y \in A$. Assuming dv to be non-zero we can compute the degrees:

$$|x| + |y| = |xy| = |dv| = |v| + 1 = n + 1.$$

As A is 1-reduced we have $|x|, |y| \geq 2$ and so by the above $|x|, |y| \leq n - 1$. Conclude that $d(V(k)) \subset \Lambda(V(n-1))$. □

Minimal models admit very nice homotopy groups. Note that for a minimal algebra ΛV there is a natural augmentation and the the differential is decomposable. Hence $Q\Lambda V$ is naturally isomorphic to $(V, 0)$. In particular the homotopy groups are simply given by $\pi^n(\Lambda V) = V^n$.

Definition 6.0.4 is the same as in [FHT01] without assuming connectivity. We find some different definitions of (minimal) Sullivan algebras in the literature. For example we find a definition using well orderings in [H+07]. The decomposability of d also admits a different characterization (at least in the connected case). The equivalence of the definitions is expressed in the following two lemmas. The first can be easily proven by choosing subspaces with bases $V'_k = \langle v_j \rangle_{j \in J_k}$ such that $V(k) = V(k-1) \oplus V'_k$ for each degree. Then choose some well order on J_k to define a well order on $J = \bigcup_k J_k$. The second lemma is a more refined version of Lemma 6.0.6. Since we will not need these equivalent definitions, the details are left out.

Lemma 6.0.7. *A cdga $(\Lambda V, d)$ is a Sullivan algebra if and only if there exists a well order J such that V is generated by v_j for $j \in J$ and $dv_j \in \Lambda V_{<j}$.*

Lemma 6.0.8. *Let $(\Lambda V, d)$ be a Sullivan algebra with $V^0 = 0$, then d is decomposable if and only if there is a well order J as above such that $i < j$ implies $|v_i| \leq |v_j|$.*

It is clear that induction will be an important technique when proving things about (minimal) Sullivan algebras. We will first prove that minimal models always exist for 1-connected cdga's and afterwards prove uniqueness.

6.1 EXISTENCE

Theorem 6.1.1. *Let (A, d) be a 1-connected cdga, then it has a minimal model $(\Lambda V, d)$.*

Proof. We construct the model and by induction on the degree. The resulting filtration will be on degree, so that the minimality follows from Lemma 6.0.6. We start by setting $V^0 = V^1 = 0$

and $V^2 = H^2(A)$. At this stage the differential is trivial, i.e. $d(V^2) = 0$. Sending the cohomology classes to their representatives extends to a map of cdga's $m_2 : \Lambda V^{\leq 2} \to A$.

Suppose $m_k : \Lambda V^{\leq k} \to A$ is constructed. We will add elements in degree $k+1$ and extend m_k to m_{k+1} to assert surjectivity and injectivity of $H(m_{k+1})$. Let $\{[a_\alpha]\}_{\alpha \in I}$ be a basis for the cokernel of $H(m_k) : H^{k+1}(\Lambda V^{\leq k}) \to H^{k+1}(A)$ and $b_\alpha \in A^{k+1}$ be a representing cycle for a_α. Let $\{[z_\beta]\}_{\beta \in J}$ be a basis for the kernel of $H(m_k) : H^{k+2}(\Lambda V^{\leq k}) \to H^{k+2}(A)$, note that $m_k(b_\beta)$ is a boundary, so that there are elements c_β such that $m_k(b_\beta) = dc_\beta$.

Define $V^{k+1} = \bigoplus_{\alpha \in I} \Bbbk \cdot v_\alpha \oplus \bigoplus_{\beta \in J} \Bbbk \cdot v'_\beta$ and extend d and m_{k+1} by defining

$$d(v_\alpha) = 0 \qquad d(v'_\beta) = z_\beta$$

$$m_{k+1}(v_\alpha) = b_\alpha \qquad m_{k+1}(v'_\beta) = c_\beta$$

Now clearly $d^2 = 0$ on the generators, so this extends to a derivation on $\Lambda V^{\leq k+1}$, similarly m_{k+1} commutes with d on the generators and hence extends to a chain map.

This finished the construction of V and $m : \Lambda V \to A$. Now we will prove that $H(m)$ is an isomorphism. We will do so by proving surjectivity and injectivity by induction on k.

Start by noting that $H^i(m_2)$ is surjective for $i \leq 2$. Now assume by induction that $H^i(m_k)$ is surjective for $i \leq k$. Since $\operatorname{im} H(m_k) \subset \operatorname{im} H(m_{k+1})$ we see that $H^i(m_{k+1})$ is surjective for $i < k+1$. By construction it is also surjective in degree $k+1$. So $H^i(m_k)$ is surjective for all $i \leq k$ for all k.

For injectivity we note that $H^i(m_2)$ is injective for $i \leq 3$, since $\Lambda V^{\leq 2}$ has no elements of degree 3. Assume $H^i(m_k)$ is injective for $i \leq k+1$ and let $[z] \in \ker H^i(m_{k+1})$. Now if $|z| \leq k$ we get $[z] = 0$ by induction and if $|z| = k+2$ we get $[z] = 0$ by construction. Finally if $|z| = k+1$, then we write $z = \sum \lambda_\alpha v_\alpha + \sum \lambda'_\beta v'_\beta + w$ where v_α, v'_β are the generators as above and $w \in \Lambda V^{\leq k}$. Now $dz = 0$ and so $\sum \lambda'_\beta v'_\beta + dw = 0$, so that $\sum \lambda'_\beta [z_\beta] = 0$. Since $\{[z_\beta]\}$ was a basis, we see that $\lambda'_\beta = 0$ for all β. Now by applying m_k we get $\sum \lambda_\alpha [b_\alpha] = H(m_k)[w]$, so that $\sum \lambda_\alpha [a_\alpha] = 0$ in the cokernel, recall that $\{[a_\alpha]\}$ formed a basis and hence $\lambda_\alpha = 0$ for all α. Now $z = w$ and the statement follows by induction. Conclude that $H^i(m_{k+1})$ is injective for $i \leq k+2$.

This concludes that $H(m)$ is indeed an isomorphism. So we constructed a weak equivalence $m : \Lambda V \to A$, where ΛV is minimal by Lemma 6.0.6. $\square$

Remark 6.1.2. *The previous construction will construct an r-reduced minimal model for an r-connected cdga A.*

Moreover if $H(A)$ is finite dimensional in each degree, then so is the minimal model ΛV. This follows inductively. First notice that V^2 is clearly finite dimensional. Now assume that $\Lambda V^{<k}$ is finite dimensional in each degree, then both the cokernel and kernel are, so we adjoin only finitely many elements in V^k.

6.2 UNIQUENESS

Before we state the uniqueness theorem we need some more properties of minimal models. In fact we will prove that Sullivan algebras are cofibrant. This allows us to use some general facts about model categories.

Lemma 6.2.1. *Sullivan algebras are cofibrant and the inclusions induced by the filtration are cofibrations.*

Proof. Consider the following lifting problem, where ΛV is a Sullivan algebra.

$$\begin{tikzcd} \Bbbk \arrow[r, "\eta"] \arrow[d, "\eta"] & X \arrow[d, two heads, "\simeq", "p"'] \\ \Lambda V \arrow[r, "g"] & Y \end{tikzcd}$$

By the left adjointness of Λ we only have to specify a map $\phi : V \to X$ which commutes with the differential such that $p \circ \phi = g$. We will do this by induction. Note that the induction step proves precisely that $(\Lambda V(k), d) \to (\Lambda V(k+1), d)$ is a cofibration.

- Suppose $\{v_\alpha\}$ is a basis for $V(0)$. Define $V(0) \to X$ by choosing preimages x_α such that $p(x_\alpha) = g(v_\alpha)$ (p is surjective). Define $\phi(v_\alpha) = x_\alpha$.

- Suppose ϕ has been defined on $V(n)$. Write $V(n+1) = V(n) \oplus V'$ and let $\{v_\alpha\}$ be a basis for V'. Then $dv_\alpha \in \Lambda V(n)$, hence $\phi(dv_\alpha)$ is defined and

  $$d\phi dv_\alpha = \phi d^2 v_\alpha = 0$$

  $$p\phi dv_\alpha = g dv_\alpha = dg v_\alpha.$$

  Now ϕdv_α is a cycle and $p\phi dv_\alpha$ is a boundary of gv_α. By the following lemma there is a $x_\alpha \in X$ such that $dx_\alpha = \phi dv_\alpha$ and $px_\alpha = gv_\alpha$. The former property proves that ϕ is a chain map, the latter proves the needed commutativity $p \circ \phi = g$.

□

Lemma 6.2.2. *Let $p : X \to Y$ be a trivial fibration, $x \in X$ a cycle, $p(x) \in Y$ a boundary of $y' \in Y$. Then there is a $x' \in X$ such that*

$$dx' = x \quad \text{and} \quad px' = y'.$$

Proof. We have $p^*[x] = [px] = 0$, since p^* is injective we have $x = d\overline{x}$ for some $\overline{x} \in X$. Now $p\overline{x} = y' + db$ for some $b \in Y$. Choose $a \in X$ with $pa = b$, then define $x' = \overline{x} - da$. Now check the requirements: $px' = p\overline{x} - pa = y'$ and $dx' = d\overline{x} - dda = d\overline{x} = x$. □

As minimal models are cofibrant Remark 4.3.6 immediately implies the following.

Corollary 6.2.3. *Let $f : X \xrightarrow{\simeq} Y$ be a weak equivalence between cdga's and M a minimal algebra. Then f induces an bijection:*

$$f_* : [M, X] \xrightarrow{\cong} [M, Y].$$

Lemma 6.2.4. *Let $\phi : (M, d) \xrightarrow{\simeq} (M', d')$ be a weak equivalence between minimal algebras. Then ϕ is an isomorphism.*

Proof. Since both M and M' are minimal, they are cofibrant and so the weak equivalence is a strong homotopy equivalence (Corollary 4.3.5). And so the induced map $\pi^n(\phi) : \pi^n(M) \to \pi^n(M')$ is an isomorphism (Lemma 4.5.2).

Since M (resp. M') is free as a cga's, it is generated by some graded vector space V (resp. V'). By an earlier remark the homotopy groups were easy to calculate and we conclude that ϕ induces an isomorphism from V to V':

$$\pi^*(\phi) : V \xrightarrow{\cong} V'.$$

By induction on the degree one can prove that ϕ is surjective and hence it is a fibration. By the lifting property we can find a right inverse ψ, which is then injective and a weak equivalence. Now the above argument also applies to ψ and so ψ is surjective. Conclude that ψ is an isomorphism and ϕ, being its right inverse, is an isomorphism as well. □

Theorem 6.2.5. *Let $m : (M, d) \xrightarrow{\simeq} (A, d)$ and $m' : (M', d') \xrightarrow{\simeq} (A, d)$ be two minimal models for A. Then there is an isomorphism $\phi(M, d) \xrightarrow{\cong} (M', d')$ such that $m' \circ \phi \sim m$.*

Proof. By Corollary 6.2.3 we have $[M', M] \cong [M', A]$. By going from right to left we get a map $\phi : M' \to M$ such that $m' \circ \phi \sim m$. On homology we get $H(m') \circ H(\phi) = H(m)$, proving that (2-out-of-3) ϕ is a weak equivalence. The previous lemma states that ϕ is then an isomorphism. $\square$

The assignment to X of its minimal model $M_X = (\Lambda V, d)$ can be extended to morphisms. Let X and Y be two cdga's and $f : X \to Y$ be a map. By considering their minimal models we get the following diagram.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ M_X \arrow[u, "m_X"', "\simeq"] \arrow[ur, "fm_X"] & M_Y \arrow[u, "m_Y"', "\simeq"] \end{tikzcd}$$

Now by Corollary 6.2.3 we get a bijection ${m_Y}_*^{-1} : [M_X, Y] \cong [M_X, M_Y]$. This gives a map $M(f) = {m_Y}_*^{-1}(fm_X)$ from M_X to M_Y. Of course this does not define a functor of cdga's as it is only well defined on homotopy classes. However it is clear that it does define a functor on the homotopy category of cdga's.

Corollary 6.2.6. *The assignment $X \mapsto M_X$ defines a functor $M : \mathbf{Ho}(\mathbf{CDGA}_{\mathbb{k},1}) \to \mathbf{Ho}(\mathbf{CDGA}_{\mathbb{k},1})$. Moreover, since the minimal model is weakly equivalent, M gives an equivalence of categories:*

$$M : \mathbf{Ho}(\mathbf{CDGA}_{\mathbb{k},1}) \cong \mathbf{Ho}(\textit{Minimal algebras}_1)$$

6.3 THE MINIMAL MODEL OF THE SPHERE

We know from singular cohomology that the cohomology ring of a n-sphere is $\mathbb{Z}[X]/(X^2)$, i.e. the cga with 1 generator X in degree n such that $X^2 = 0$. This allows us to construct a minimal model for S^n.

Definition 6.3.1. Define $A(n)$ to be the cdga defined as

$$A(n) = \begin{cases} \Lambda(e) \quad |e| = n \quad de = 0 & \text{if } n \text{ is odd} \\ \Lambda(e, f) \quad |e| = n, |f| = 2n-1 \quad df = e^2 & \text{if } n \text{ is even} \end{cases}.$$

To prove that this indeed defines minimal models, we first note that $A(n)$ indeed has the same cohomology groups. All we need to prove is that there is an actual weak equivalence $A(n) \to A(S^n)$.

For the odd case, we can choose a representative $y \in A(S^n)$ for the generator X. Sending e to y defines a map $\phi : A(n) \to$

$A(S^n)$. Note that since $|y|$ is odd we have $y^2 = 0$ by commutativity of $A(S^n)$, so indeed ϕ is a map of algebras. Both e and y are cocycles, so ϕ is a chain map. Finally we see that $H(\phi)$ sends $[e]$ to X, hence this is an isomorphism.

For the even case, we need to choose two elements in $A(S^n)$. Again let $y \in A(S^n)$ be a representative for X. Now since $X^2 = 0$ there is an element $c \in A(S^n)$ such that $y^2 = dc$. Sending e to y and f to c defines a map of cdga's $\phi : A(n) \to A(S^n)$. And $H(\phi)$ sends the class $[e]$ to X.

7 THE MAIN EQUIVALENCE

In this chapter we aim to prove that the homotopy theory of rational spaces is the same as the homotopy theory of cdga's over $\mathbb{k} = \mathbb{Q}$. We will only work over the rationals in this chapter, so we will omit the subscript $\mathbb{Q}$ in many places. Before we prove the equivalence, we will show that A and K form a Quillen pair. This already provides an adjunction between the homotopy categories. Besides the equivalence of the homotopy categories we will also prove that the homotopy groups of a space will be dual to the homotopy groups of the associated cdga.

We will prove that A preserves cofibrations and trivial cofibrations. We only have to check this fact for the generating (trivial) cofibrations in **sSet**. Note that the contravariance of A means that a (trivial) cofibrations should be sent to a (trivial) fibration.

Lemma 7.0.2. *$A(i) : A(\Delta[n]) \to A(\partial\Delta[n])$ is surjective.*

Proof. Let $\phi \in A(\partial\Delta[n])$ be an element of degree k, hence it is a map $\partial\Delta[n] \to {A_{PL}}^k$. We want to extend this to the whole simplex. By the fact that ${A_{PL}}^k$ is a Kan complex and contractible we can find a lift $\overline{\phi}$ in the following diagram showing the surjectivity.

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r, "\phi"] \arrow[d, hook, "i"] & {A_{PL}}^k \\
\Delta[n] \arrow[ur, dashed, "\overline{\phi}"'] &
\end{tikzcd}$$

□

Lemma 7.0.3. *$A(j) : A(\Delta[n]) \to A(\Lambda^k_n)$ is surjective and a quasi isomorphism.*

Proof. As above we get surjectivity from the Kan condition. To prove that $A(j)$ is a quasi isomorphism we pass to the singular cochain complex and use that $C^*(j) : C^*(\Delta[n]) \xrightarrow{\simeq} C^*(\Lambda^n_k)$

is a quasi isomorphism. Consider the following diagram and conclude that $A(j)$ is surjective and a quasi isomorphism.

$$\begin{tikzcd} A(\Delta[n]) \arrow[r, "A(j)"] \arrow[d, "\wr\oint"] & A(\Lambda^k_n) \arrow[d, "\wr\oint"] \\ C^*(\Delta[n]) \arrow[r, "C^*(j)"] & C^*(\Lambda^k_n) \end{tikzcd}$$

□

Since A is a left adjoint, it preserves all colimits and by functoriality it preserves retracts. From this we can conclude the following corollary.

Corollary 7.0.4. *A preserves all cofibrations and all trivial cofibrations and hence is a left Quillen functor.*

Corollary 7.0.5. *A and K induce an adjunction on the homotopy categories:*

$$LA : \mathbf{Ho}(\mathbf{sSet}) \rightleftarrows \mathbf{Ho}(\mathbf{CDGA})^{op} : RK.$$

The induced adjunction in the previous corollary is given by $LA(X) = A(X)$ for $X \in \mathbf{sSet}$ (note that every simplicial set is already cofibrant) and $RK(Y) = K(Y^{cof})$ for $Y \in \mathbf{CDGA}$. By the use of minimal models, and in particular the functor M. We get the following adjunction between 1-connected objects:

Corollary 7.0.6. *There is an adjunction:*

$$M : \mathbf{Ho}(\mathbf{sSet}_1) \rightleftarrows \mathbf{Ho}(\textit{Minimal models}_1)^{op} : RK,$$

where M is given by $M(X) = M(A(X))$ and RK is given by $RK(Y) = K(Y)$ (note that minimal models are always cofibrant).

7.1 HOMOTOPY GROUPS

The homotopy groups of augmented cdga's are precisely the dual of the homotopy groups of their associated spaces.

Theorem 7.1.1. *Let X be a cofibrant augmented cdga, then there is a natural bijection*

$$\pi_n(KX) \cong \pi^n(X)^*.$$

Proof. We will prove the following chain of natural isomorphisms:

$$\begin{aligned}\pi_n(KX) = [S^n, KX] &\overset{(1)}{\cong} [X, A(S^n)] \\ &\overset{(2)}{\cong} [X, A(n)] \overset{(3)}{\cong} [X, V(n)] \overset{(4)}{\cong} \pi^n(X)^*\end{aligned}$$

The first isomorphism (1) follows from the homotopy adjunction in Corollary 7.0.5 (note that KX is a Kan complex, since it is a simplicial group). The next two isomorphisms (2) and (3) are induced by the weak equivalences $A(n) \xrightarrow{\simeq} A(S^n)$ and $A(n) \xrightarrow{\simeq} V(n)$ by using Corollary 4.3.5. Finally we get (4) from Lemma 4.5.3. □

Theorem 7.1.2. *Let X be a 1-connected cofibrant augmented cdga, then the above bijection is a group isomorphism $\pi_n(KX) \cong \pi^n(X)^*$.*

Proof. We will prove that the map in the previous theorem preserves the group structure. We will prove this by endowing a certain cdga with a coalgebra structure, which induces the multiplication in both $\pi_n(KX)$ and $\pi^n(X)^*$.

Since every 1-connected cdga admits a minimal model, we will assume that X is a minimal model, generated by V (filtered by degree). We first observe that $\pi^n(X) \cong \pi^n(\Lambda V^{\leq n}$, since elements of degree $n+1$ or higher do not influence the homology of QX.

Now consider the cofibration $i : \Lambda V(n-1) \hookrightarrow \Lambda V(n)$ and its associated long exact sequence (Corollary 4.5.5). It follows that $\pi^n(\Lambda V(n)) \cong \pi^n(\mathrm{coker}(i))$. Now $\mathrm{coker}(i)$ is generated by elements of degree n only (as algebra), i.e. $\mathrm{coker}(i) = (\Lambda W, 0)$ for some vector space $W = W^n$. Let Y denote this space $Y = (\Lambda W, 0)$. Define a comultiplication on generators $w \in W$:

$$\Delta : Y \to Y \otimes Y : \quad w \mapsto 1 \otimes w + w \otimes 1.$$

This will always define a map on free cga's, but in general might not respect the differential. But since the differential is trivial, this defines a map of cdga's. We have the following diagram:

$$\begin{tikzcd}
\pi_n(KY) \times \pi_n(KY) \arrow[d, "\cong"] \arrow[dr, "\mu"] & \\
\pi_n(KY \times KY) \arrow[r, "\Delta^*"] \arrow[d, dashed, "\cong"] & \pi_n(KY) \arrow[d, dashed, "\cong"] \\
\pi^n(Y \otimes Y)^* \arrow[r, "\Delta^*"] \arrow[d, "\cong"] & \pi^n(Y)^* \\
\pi^n(Y)^* \oplus \pi^n(Y)^* \arrow[ur, "+"'] &
\end{tikzcd}$$

The middle part commutes by the naturality of the isomorphisms described in Theorem 7.1.1. The only thing we need to prove is that the upper triangle and bottom triangle commute.

For the upper triangle we note that KY is in now a simplicial monoid (induced by the map Δ), and we know from [GJ99] that the multiplication on the homotopy groups of a simplicial monoid is induced by the monoid structure.

For the bottom triangle we first note that the isomorphism $\pi^n(Y) \oplus \pi^n(Y) \cong \pi^n(Y \otimes Y)$ follows from Lemma 4.4.3. Now the induced map $QY \xrightarrow{Q\Delta} Q(Y \otimes Y) \cong Q(Y) \oplus QY$ is defined as $w \mapsto (w, w)$, and so the dual is precisely addition.

So the multiplication on the homotopy groups $\pi_n(KY)$ and $\pi^n(Y)^*$ are induced by the same map. So by the commutativity of the above diagram the natural bijection is a group isomorphism. $\square$

Recall that for a minimal model $M = (\Lambda V, d)$ the homotopy groups equal $\pi^n(M) = V^n$. So in particular we know the homotopy groups of the space KM.

Corollary 7.1.3. *Let $M = (\Lambda V, d)$ be a minimal algebra, then the homotopy groups of KM are $\pi_n(KM) = V^{n*}$.*

In particular, for a cdga with only one generator $M = \Lambda(v)$ with $dv = 0$ and $|v| = n$, we conclude that KM is an Eilenberg-MacLane space of type $K(\mathbb{k}^*, n)$.

7.2 THE EILENBERG-MOORE THEOREM

Before we prove the actual equivalence, we will discuss a theorem of Eilenberg and Moore. The theorem tells us that the singular cochain complex of a pullback along a fibration is nice in a particular way. The theorem and its proof (using spectral sequences) can be found in [McC01, Theorem 7.14].

Theorem 7.2.1. *Given the following pullback diagram of spaces*

$$\begin{tikzcd} E_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

where p is a fibration, all spaces are 0-connected and B is 1-connected. The cohomology with coefficients in a field $\mathbb{k}$ can be computed by an isomorphism

$$\operatorname{Tor}_{C^*(B;\mathbb{k})}(C^*(X;\mathbb{k}), C^*(E;\mathbb{k})) \xrightarrow{\cong} H(C^*(E_f;\mathbb{k})).$$

Now the Tor group appearing in the theorem can be computed via a *bar construction*. The explicit construction for cdga's can be found in [BG76], but also in [Ols] where it is related to the homotopy pushout of cdga's. We will not discuss the details of the bar construction. However it is important to know that the Tor group only depends on the cohomology of the dga's in use (see [McC01, Corollary 7.7]), in other words: quasi isomorphic dga's (in a compatible way) will have isomorphic Tor groups. Since $C^*(-;\mathbb{Q})$ is isomorphic to $A(-)$, the above theorem also holds for our functor A. We can restate the theorem as follows.

Corollary 7.2.2. *Given the following pullback diagram of spaces*

$$\begin{tikzcd} E_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

where p is a fibration. Assume that all spaces are 0-connected and B is 1-connected. Then the induced diagram

$$\begin{tikzcd} A(B) \arrow[r] \arrow[d] & A(E) \arrow[d] \\ A(X) \arrow[r] & A(E_f) \end{tikzcd}$$

is a homotopy pushout.

Another exposition of this corollary can be found in [Ber12, Section 8.4]. A very brief summary of the above statement is that A sends homotopy pullbacks to homotopy pushout (assuming some connectedness).

7.3 EQUIVALENCE ON RATIONAL SPACES

In this section we will prove that the adjunction in Corollary 7.0.6 is in fact an equivalence when restricted to certain subcategories. One of the restrictions is the following.

Definition 7.3.1. A cdga A is said to be of *finite type* if $H(A)$ is finite dimensional in each degree. Similarly X is of *finite type* if $H^i(X;\mathbb{Q})$ is finite dimensional for each i.

Note that X is of finite type if and only if $A(X)$ is of finite type.

For the equivalence of rational spaces and cdga's we need that the unit and counit of the adjunction in Corollary 7.0.6 are in fact weak equivalences for rational spaces. More formally: for any (automatically cofibrant) $X \in \mathbf{sSet}$ and any minimal model $A \in \mathbf{CDGA}$, both rational, 1-connected and of finite type, the following two natural maps are weak equivalences:

$$X \to K(M(X))$$
$$A \to M(K(A))$$

where the first of the two maps is given by the composition $X \to K(A(X)) \xrightarrow{K(m_X)} K(M(X))$, and the second map is obtained by the map $A \to A(K(A))$ and using the bijection from Corollary 6.2.3: $[A, A(K(A))] \cong [A, M(K(A))]$. By the 2-out-of-3 property the map $A \to M(K(A))$ is a weak equivalence if and only if the ordinary unit $A \to A(K(A))$ is a weak equivalence.

Lemma 7.3.2. *(Base case) Let $A = (\Lambda(v), 0)$ be a minimal model with one generator of degree $|v| = n \geq 1$. Then $A \xrightarrow{\simeq} A(K(A))$.*

Proof. By Corollary 7.1.3 we know that $K(A)$ is an Eilenberg-MacLane space of type $K(\mathbb{Q}^*, n)$. The cohomology of an Eilenberg-MacLane space with coefficients in $\mathbb{Q}$ is known (note that this is specific for $\mathbb{Q}$):

$$H^*(K(\mathbb{Q}^*, n); \mathbb{Q}) = \mathbb{Q}[x],$$

that is, the free commutative graded algebra with one generator x. This can be calculated, for example, with spectral sequences [GM13].

Now choose a cycle $z \in A(K(\mathbb{Q}^*, n))$ representing the class x and define a map $A \to A(K(A))$ by sending the generator v to z. This induces an isomorphism on cohomology. So A is the minimal model for $A(K(A))$. $\square$

Lemma 7.3.3. *(Induction step) Let A be a cofibrant, connected algebra. Let B be the pushout in the following square, where $m \geq 1$:*

$$\begin{tikzcd} \Lambda S(m+1) \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\ \Lambda D(m) \arrow[r] & B \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Then if $A \to A(K(A))$ is a weak equivalence, so is $B \to A(K(B))$.

Proof. Applying K to the above diagram gives a pullback diagram of simplicial sets, where the induced vertical maps are fibrations (since K is right Quillen). In other words, the induced square is a homotopy pullback.

Applying A again gives the following cube of cdga's:

$$\begin{tikzcd}
\Lambda S(m+1) \arrow[rr] \arrow[dd, hook] \arrow[dr, "\simeq"] & & A \arrow[dd, hook] \arrow[dr, "\simeq"] & \\
& A(K(\Lambda S(m+1))) \arrow[rr] \arrow[dd] & & A(K(A)) \arrow[dd] \\
\Lambda D(m) \arrow[rr] \arrow[dr, "\simeq"] & & B \arrow[dr] & \\
& A(K(\Lambda D(m))) \arrow[rr] & & A(K(B))
\end{tikzcd}$$

Note that we have a weak equivalence in the top left corner, by the base case ($\Lambda S(m+1) = (\Lambda(v), 0)$). The weak equivalence in the top right is by assumption. Finally the bottom left map is a weak equivalence because both cdga's are acyclic.

By Corollary 7.2.2 we know that the front face is a homotopy pushout. The back face is a homotopy pushout by Lemma B.4.1 and to conclude that $B \to A(K(B))$ is a weak equivalence, we use the cube lemma (Lemma B.4.2). $\square$

Now we wish to use the previous lemma as an induction step for minimal models. Let $(\Lambda V, d)$ be some minimal algebra. Write $V(n+1) = V(n) \oplus V'$ and let $v \in V'$ of degree $|v| = k$, then the minimal algebra $(\Lambda(V(n) \oplus \mathbb{Q} \cdot v), d)$ is the pushout in the following diagram, where f sends the generator c to dv.

$$\begin{tikzcd}
S(k) \arrow[r, "f"] \arrow[d, hook] & (\Lambda V(n), d) \arrow[d] \\
T(k-1) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & (\Lambda(V(n) \oplus \mathbb{Q} \cdot v), d)
\end{tikzcd}$$

In particular if the vector space V' is finitely generated, we can repeat this procedure for all basis elements (it does not matter in what order we do so, as $dv \in \Lambda V(n)$). So in this case where V' is finite-dimensional, if $(\Lambda V(n), d) \to A(K(\Lambda V(n), d))$ is a weak equivalence, then by the above lemma $(\Lambda V(n+1), d) \to A(K(\Lambda V(n+1), d))$ is a weak equivalence as well.

Note that by Remark 6.1.2 every cdga of finite type has a minimal model which is finite dimensional in each degree.

Corollary 7.3.4. *Let $(\Lambda V, d)$ be a 1-connected minimal algebra with V^i finite dimensional for all i. Then $(\Lambda V, d) \to A(K(\Lambda V, d))$ is a weak equivalence.*

Proof. Note that if we want to prove the isomorphism $H^i(\Lambda V, d) \to H^i(A(K(\Lambda V, d)))$ it is enough to prove that $H^i(\Lambda V^{\leq i}, d) \to H^i(A(K(\Lambda V^{\leq i}, d)))$ is an isomorphism (as the elements of higher degree do not change the isomorphism). By the 1-connectedness we can choose our filtration to respect the degree by Lemma 6.0.6.

Now $V(n)$ is finitely generated for all n by assumption. By the inductive procedure above we see that $(\Lambda V(n), d) \to A(K(\Lambda V(n), d))$ is a weak equivalence for all n. Hence $(\Lambda V, d) \to A(K(\Lambda V, d))$ is a weak equivalence. □

Now we want to prove that $X \to K(M(X))$ is a weak equivalence for a simply connected rational space X of finite type. For this, we will use that A preserves and detects such weak equivalences by the Serre-Whitehead theorem (Corollary 2.1.4). To be precise: for a simply connected rational space X the map $X \to K(M(X))$ is a weak equivalence if and only if $A(K(M(X))) \to A(X)$ is a weak equivalence.

Lemma 7.3.5. *The map $X \to K(M(X))$ is a weak equivalence for 1-connected, rational spaces X of finite type.*

Proof. Recall that the map $X \to K(M(X))$ was defined to be the composition of the actual unit of the adjunction and the map $K(m_X)$. When applying A we get the following situation, where commutativity is ensured by the adjunction laws:

$$\begin{tikzcd}
A(X) & A(K(A(X))) \arrow[l] & A(K(M(X))) \arrow[l] \\
 & A(X) \arrow[ul, "\mathbf{id}"] \arrow[u] & M(X) \arrow[l, "\simeq"'] \arrow[u]
\end{tikzcd}$$

The map on the right is a weak equivalence by Corollary 7.3.4. Then by the 2-out-of-3 property we see that the above composition is indeed a weak equivalence. Since A detects weak equivalences, we conclude that $X \to K(M(X))$ is a weak equivalence. □

We have proven the following theorem.

Theorem 7.3.6. *The functors A and K induce an equivalence of homotopy categories, when restricted to rational, 1-connected objects of finite type. More formally, we have:*

$$\mathbf{Ho}(\mathbf{sSet}_{\mathbb{Q},1,f}) \cong \mathbf{Ho}(\mathbf{CDGA}_{\mathbb{Q},1,f}).$$

Furthermore, for any 1-connected space X of finite type, we have the following isomorphism of groups:

$$\pi_i(X) \otimes \mathbb{Q} \cong V^{i^*},$$

where $(\Lambda V, d)$ is the minimal model of $A(X)$.

Finally we see that for a 1-connected space X of finite type, we have a natural rationalization:

$$X \to K(A(X))$$

Part III

APPLICATIONS AND FURTHER TOPICS

8 RATIONAL HOMOTOPY GROUPS OF THE SPHERES AND OTHER CALCULATIONS

In this chapter we will calculate the rational homotopy groups of the spheres using minimal models. The minimal model for the sphere was already given, but we will quickly redo the calculation.

8.1 THE SPHERE

Proposition 8.1.1. *For odd n the rational homotopy groups of S^n are given by*

$$\pi_i(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & \text{if } i = n \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We know the cohomology of the sphere by classical results:

$$H^i(S^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} \cdot 1, & \text{if } i = 0 \\ \mathbb{Q} \cdot x, & \text{if } i = n \\ 0, & \text{otherwise,} \end{cases}$$

where x is a generator of degree n. Define $M_{S^n} = \Lambda(e)$ with $d(e) = 0$ and e of degree n. Notice that since n is odd, we get $e^2 = 0$. By taking a representative for x, we can give a map $M_{S^n} \to A(S^n)$, which is a weak equivalence.

Clearly M_{S^n} is minimal, and hence it is a minimal model for S^n. By Theorem 7.3.6 we have

$$\pi_*(S^n) \otimes \mathbb{Q} = \pi_*(K(M_{S^n})) = \pi^*(M_{S^n})^* = \mathbb{Q} \cdot e^*.$$

□

Proposition 8.1.2. *For even n the rational homotopy groups of S^n are given by*

$$\pi_i(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & \text{if } i = 2n-1 \\ \mathbb{Q}, & \text{if } i = n \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Again since we know the cohomology of the sphere, we can construct its minimal model. Define $M_{S^n} = \Lambda(e, f)$ with $d(e) = 0, d(f) = e^2$ and $|e| = n, |f| = 2n - 1$. Let $[x] \in H^n(S^n; \mathbb{Q})$ be a generator and $x \in A(S^n)$ its representative, then notice that $[x]^2 = 0$. This means that there exists an element $y \in A(S^n)$ such that $dy = x^2$. Mapping e to x and f to y defines a quasi isomorphism $M_{S^n} \to A(S^n)$.

Again we can use Corollary 7.1.3 to directly conclude:

$$\pi_*(S^n) \otimes \mathbb{Q} = \pi^*(M_{S^n})^* = \mathbb{Q} \cdot e^* \oplus \mathbb{Q} \cdot f^*.$$

□

The generators e and f in the last proof are related by the so called *Whitehead product*. The whitehead product is a bilinear map $\pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X)$ satisfying a graded commutativity relation and a graded Jacobi relation, see [FHT01]. If we define a *Whitehead algebra* to be a graded vector space with such a map satisfying these relations, we can summarize the above two propositions as follows [Ber12].

Corollary 8.1.3. *The rational homotopy groups of S^n are given by*

$$\pi_*(S^n) \otimes \mathbb{Q} = \textit{the free whitehead algebra on 1 generator.}$$

Together with the fact that all groups $\pi_i(S^n)$ are finitely generated (this is proven by Serre in [Ser53]) we can conclude that $\pi_i(S^n)$ is a finite group unless $i = n$ and unless $i = 2n - 1$ for even n. The fact that $\pi_i(S^n)$ are finitely generated can be proven by the Serre-Hurewicz theorems (Theorem 2.0.9) when taking the Serre class of finitely generated abelian groups (but this requires a weaker notion of a Serre class, and stronger theorems, than the one given in this thesis).

8.2 EILENBERG-MACLANE SPACES

The following result is already used in proving the main theorem. But using the main theorem it is an easy and elegant consequence.

Proposition 8.2.1. *For an Eilenberg-MacLane space of type $K(\mathbb{Z}, n)$ we have:*

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x],$$

i.e. the free graded commutative algebra on 1 generator.

Proof. By the existence theorem for minimal models, we know there is a minimal model $(\Lambda V, d) \xrightarrow{\simeq} A(K(\mathbb{Z}, n))$. By calculating the homotopy groups we see

$$V^{i*} = \pi^i(\Lambda V)^* = \pi_i(K(\mathbb{Z}, n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & \text{if } i = n \\ 0, & \text{otherwise.} \end{cases}$$

This means that V is concentrated in degree n and that the differential is trivial. Take a generator x of degree n such that $V = \mathbb{Q} \cdot x$ and conclude that the cohomology of the minimal model, and hence the rational cohomology of $K(\mathbb{Z}, n)$, is $H(\Lambda V, 0) = \mathbb{Q}[x]$. □

Note that both the result on the spheres and this result are very different in ordinary homotopy theory. The ordinary homotopy groups of the spheres are very hard to calculate and in many cases even unknown. Similarly the (co)homology of Eilenberg-MacLane spaces is complicated (but known). In rational homotopy theory, both are easy to calculate.

Another remarkable thing happens here, the odd spheres are rationally equivalent to Eilenberg-MacLane spaces. In a further section we will briefly see that this allows us to prove that $S^n_{\mathbb{Q}}$ is an H-space if and only if n is odd.

8.3 PRODUCTS

Let X and Y be two 1-connected spaces, we will determine the minimal model for $X \times Y$. We have the two projections maps $X \times Y \xrightarrow{\pi_X} X$ and $X \times Y \xrightarrow{\pi_Y} Y$ which induces maps of cdga's: $A(X) \xrightarrow{\pi_{X*}} A(X \times Y)$ and $A(Y) \xrightarrow{\pi_{Y*}} A(X \times Y)$. Because we are working with commutative algebras, we can multiply the two maps to obtain:

$$\mu : A(X) \otimes A(Y) \xrightarrow{\pi_{X*} \cdot \pi_{Y*}} A(X \times Y).$$

This is different from the singular cochain complex where the Eilenberg-Zilber map is needed. By passing to cohomology the multiplication is identified with the cup product. Hence, by applying the Künneth theorem, we see that μ is a weak equivalence.

Now if $M_X = (\Lambda V, d_X)$ and $M_Y = (\Lambda W, d_Y)$ are the minimal models for $A(X)$ and $A(Y)$, we see that $M_X \otimes M_Y \xrightarrow{\simeq} A(X) \otimes A(Y)$ is a weak equivalence (again by the Künneth theorem). Furthermore $M_X \otimes M_Y = (\Lambda V \otimes \Lambda W, d_X \otimes d_Y)$ is itself minimal, with $V \oplus W$ as generating space. As a direct consequence we see that

$$\begin{aligned}\pi_i(X \times Y) \otimes \mathbb{Q} &\cong \pi^i(M_X \otimes M_Y)^* \\ &\cong V^{i^*} \oplus W^{i^*} \cong \pi_i(X) \oplus \pi_i(Y),\end{aligned}$$

which of course also follows from the classical result that ordinary homotopy groups already commute with products [May99].

Going from cdga's to spaces is easier. Since K is a right adjoint from $\mathbf{CDGA}^{\mathrm{op}}$ to $\mathbf{sSet}$ it preserves products. For two (possibly minimal) cdga's A and B, this means:

$$K(A \otimes B) \cong K(A) \times K(B).$$

Since the geometric realization of simplicial sets also preserve products, we get

$$|K(A \otimes B)| \cong |K(A)| \times |K(B)|.$$

8.4 H-SPACES

In this section we will prove that the rational cohomology of an H-space is free as commutative graded algebra. We will also give its minimal model and relate it to the homotopy groups. In some sense H-spaces are homotopy generalizations of topological monoids. In particular topological groups (and hence Lie groups) are H-spaces.

Definition 8.4.1. An *H-space* is a pointed topological space $x_0 \in X$ with a map $\mu : X \times X \to X$, such that $\mu(x_0, -), \mu(-, x_0) : X \to X$ are homotopic to $\mathbf{id}_X$.

Let X be an 0-connected H-space of finite type, then we have the induced comultiplication map

$$\mu^* : H^*(X; \mathbb{Q}) \to H^*(X; \mathbb{Q}) \otimes H^*(X; \mathbb{Q}).$$

Homotopic maps are sent to equal maps in cohomology, so we get $H^*(\mu(x_0, -)) = \mathbf{id}_{H^*(X;\mathbb{Q})}$. Now write $H^*(\mu(x_0, -)) = (\epsilon \otimes \mathbf{id}) \circ H^*(\mu)$, where ϵ is the augmentation induced by x_0, to conclude that for any $h \in H^+(X; \mathbb{Q})$ the image is of the form

$$H^*(\mu)(h) = h \otimes 1 + 1 \otimes h + \psi,$$

for some element $\psi \in H^+(X; \mathbb{Q}) \otimes H^+(X; \mathbb{Q})$. This means that the comultiplication is counital.

Choose a subspace V of $H^+(X;\mathbb{Q})$ such that $H^+(X;\mathbb{Q}) = V \oplus H^+(X;\mathbb{Q}) \cdot H^+(X;\mathbb{Q})$. In particular we get $V^1 = H^1(X;\mathbb{Q})$ and $H^2(X;\mathbb{Q}) = V^2 \oplus H^1(X;\mathbb{Q}) \cdot H^1(X;\mathbb{Q})$. Continuing with induction we see that the induced map $\phi : \Lambda V \to H^*(X;\mathbb{Q})$ is surjective. One can prove (by induction on the degree and using the counitality) that the elements in V are primitive, i.e. $\mu^*(v) = 1 \otimes v + v \otimes 1$ for all $v \in V$. The free algebra also admits a comultiplication, by requiring that the generators are the primitive elements. It follows that the following diagram commutes:

$$\begin{tikzcd}
\Lambda V \arrow[r, "\phi"] \arrow[d, "\Delta"] & H^*(X;\mathbb{Q}) \arrow[d, "\mu^*"] \\
\Lambda V \otimes \Lambda V \arrow[r, "\phi \otimes \phi"] & H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})
\end{tikzcd}$$

We will now prove that ϕ is also injective. Suppose by induction that ϕ is injective on $\Lambda V^{<n}$. An element $w \in \Lambda V^{\leq n}$ can be written as $\sum_{k_1,\ldots,k_r} v_1^{k_1} \cdots v_r^{k_r} a_{k_1 \cdots k_r}$, where $\{v_1, \ldots, v_r\}$ is a basis for V^n and $a_{k_1 \cdots k_r} \in \Lambda V^{<n}$. Assume $\phi(w) = 0$. Let $\pi : H^*(X;\mathbb{Q}) \to H^*(X;\mathbb{Q})/\phi(\Lambda V^{<n})$ be the (linear) projection map. Now consider the image of $(\pi \otimes \mathbf{id})\mu^*(\phi(w)$ in the component $\mathrm{im}(\pi)^n \otimes H^*(X;\mathbb{Q})$, it can be written as (here we use the above commuting square):

$$\sum_i (\pm \pi(v_i) \otimes \phi(\sum_{k_1,\ldots,k_r} k_i v_1^{k_1} \cdots v_i^{k_i - 1} \cdots v_r^{k_r} a_{k_1 \cdots k_r})$$

As $\phi(w) = 0$ and the elements $\pi(v_i)$ are linearly independent, we see that $\phi(\sum_{k_1,\ldots,k_r} k_i v_1^{k_1} \cdots v_i^{k_i-1} \cdots v_r^{k_r} a_{k_1 \cdots k_r}) = 0$ for all i. By induction on the degree of w (the base case being $|w| = n$ is trivial), we conclude that

$$\sum_{k_1,\ldots,k_r} k_i v_1^{k_1} \cdots v_i^{k_i-1} \cdots v_r^{k_r} a_{k_1 \cdots k_r} = 0 \text{ for all } i$$

This means that either all $k_i = 0$, in which case $w \in \Lambda V^{<n}$ and so $w = 0$ by induction, or all $a_{k_1,\ldots,k_r} = 0$, in which case we have $w = 0$. This proves that ϕ is injective.

We have proven that $\phi : \Lambda V \to H^*(H;\mathbb{Q})$ is an isomorphism. So the cohomology of an H-space is free as cga. Now we can choose cocycles in $A(X)$ which represent the cohomology classes. More precisely for $v_i^{(n)} \in V^n$ we choose $w_i^{(n)} \in A(X)^n$ representing it. This defines a map, which sends $v_i^{(n)}$ to $w_i^{(n)}$. Since $w_i^{(n)}$ are cocycles, this is a map of cdga's:

$$m : (\Lambda V, 0) \to A(X)$$

Now by the calculation above, this is a weak equivalence. Furthermore $(\Lambda V, 0)$ is minimal. We have proven the following lemma:

Lemma 8.4.2. *Let X be a 0-connected H-space of finite type. Then its minimal model is of the form $(\Lambda V, 0)$. In particular we see:*

$$H(X; \mathbb{Q}) = \Lambda V \qquad \pi_*(X) \otimes \mathbb{Q} = V^*$$

This allows us to directly relate the rational homotopy groups to the cohomology groups. Since the rational cohomology of the sphere S^n is not free (as algebra) when n is even we get the following.

Corollary 8.4.3. *The spheres S^n are not H-spaces if n is even.*

In fact we have that $S^n_{\mathbb{Q}}$ is an H-space if and only if n is odd. The only if part is precisely the above corollary. The if part follows from the fact that $S^n_{\mathbb{Q}}$ is the Eilenberg-MacLane space $K(\mathbb{Q}^*, n)$ for odd n.

9 FURTHER TOPICS

9.1 Quillen's approach

In this thesis we used Sullivan's approach to give algebraic models for rational spaces. However, Sullivan was not the first to give algebraic models. Quillen gave a dual approach in [Qui69]. By a long chain of homotopy equivalences his main result is

$$\begin{aligned}\mathbf{Ho}(\mathbf{Top}_{\mathbb{Q},1}) &\cong \mathbf{Ho}(\text{dg } \textit{Lie} \text{ algebras}_{\mathbb{Q},0}) \\ &\cong \mathbf{Ho}(\text{cdg } \textit{co}\text{algebras}_{\mathbb{Q},1})\end{aligned}$$

The first category is the one of differential graded Lie algebras over $\mathbb{Q}$ and the second is cocommutative (coassociative) differential graded coalgebras. Quillen's approach does not need the finite dimensionality assumptions and is hence more general.

Minimal models in these categories also exist, as shown in [Nei78]. They are defined analogously, we require the object to be cofibrant (or fibrant in the case of coalgebras) and that the differential is zero in the chain complex of indecomposables. Of course the meaning of indecomposable depends on the category.

Despite the generality of Quillen's approach, the author of this thesis prefers the approach by Sullivan as it provides a single, elegant functor $A : \mathbf{sSet} \to \mathbf{CDGA}$. Moreover cdga's are easier to manipulate, as commutative ring theory is a more basic subject than Lie algebras or coalgebras.

9.2 Nilpotency

In many locations in this thesis we assumed simply connectedness of objects (both spaces an cdga's). The assumption was often use to prove the base case of some inductive argument. In [BG76] the main equivalence is proven for so called nilpotent spaces (which is more general than 1-connected spaces).

In short, a nilpotent group is a group which is constructed by finitely many extensions of abelian groups. A space is called nilpotent if its fundamental group is nilpotent and the action of π_1 on π_n satisfies a related requirement.

Many theorems remain valid when assuming nilpotent spaces instead of simply connected spaces. However the proofs are complicated, as they need another inductive argument on these extensions of abelian groups in the base case.

9.3 LOCALIZATIONS AT PRIMES

In Chapter 2 we proved some results by Serre to relate homotopy groups and homology groups modulo a class of abelian groups. Now the class of p-torsion groups and the class of p-divisible groups are also Serre classes. This suggests that we can also "localize homotopy theory at primes". Indeed the construction in Chapter 3 can be altered to give a p-localization X_p of a space X. Recall that the rationalization was constructed as a telescope with a copy of the sphere for each $k > 0$. The kth copy was used in order to divid by k. For the p-localization we only add a copy of the sphere for $k > 0$ relative prime to p.

Now that we have a bunch of localizations $X_{\mathbb{Q}}, X_2, X_3, X_5, \ldots$ we might wonder what homotopical information of X we can recover from these localizations. In other words: can we go from local to global? The answer is yes in the following sense. Details can be found in [MP11] and [SR05].

Theorem 9.3.1. *Let X be a space, then X is the homotopy pullback in*

$$\begin{array}{ccc} X & \longrightarrow & \prod_{p \text{ prime}} X_p \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_{p \text{ prime}} X_p)_{\mathbb{Q}} \end{array}$$

This theorem is known as *the arithmetic square*, *fracture theorem* or *local-to-global theorem*.

As an example we find that if X is an H-space, then so are its localizations. The converse also holds when certain compatibility requirements are satisfied [SR05]. In the previous section we were able to prove that $S^n_{\mathbb{Q}}$ is an H-space if and only if n is odd. It turns out that the prime $p = 2$ brings the key to Adams' theorem: for odd n we have that S^n_2 is an H-space if and only if $n = 1, 3$ or 7. For the other primes S^n_p is always an H-space for odd n. This observation leads to one approach to prove Adams' theorem.

Part IV

APPENDICES

A

DIFFERENTIAL GRADED ALGEBRA

In this section $\Bbbk$ will be any commutative ring. We will recap some of the basic definitions of commutative algebra in a graded setting. By *linear*, *module*, *tensor product*, etc... we always mean $\Bbbk$-linear, $\Bbbk$-module, tensor product over $\Bbbk$, etc....

A.1 GRADED ALGEBRA

Definition A.1.1. A module M is said to be *graded* if it is equipped with a decomposition

$$M = \bigoplus_{n \in \mathbb{Z}} M_n.$$

An element $x \in M_n$ is called a *homogeneous element* and said to be of *degree* $|x| = n$.

If M is just any module, it always has the trivial grading given by $M_0 = M$ and $M_i = 0$ for $i \neq 0$, i.e. M is *concentrated in degree 0*. In particular $\Bbbk$ itself is a graded module concentrated in degree 0.

Definition A.1.2. A linear map $f : M \to N$ between graded modules is *graded of degree p* if it respects the grading and raises the degree by p, i.e.

$$f\big|_{M_n} : M_n \to N_{n+p}.$$

Definition A.1.3. The graded maps $f : M \to N$ between graded modules can be arranged in a graded module by defining:

$$\mathbf{Hom}_{gr}(M, N)_n = \{f : M \to N \mid f \text{ is graded of degree } n\}.$$

Note that not all linear maps can be decomposed into a sum of graded maps, so that $\mathbf{Hom}_{gr}(M, N) \subset \mathbf{Hom}(M, N)$ may be proper for some M and N.

Recall that the tensor product of modules distributes over direct sums. This defines a natural grading on the ordinary tensor product.

Definition A.1.4. The graded tensor product is defined as:

$$(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes N_j.$$

The tensor product extends to graded maps. Let $f : A \to B$ and $g : X \to Y$ be two graded maps, then their tensor product $f \otimes g : A \otimes B \to X \otimes Y$ is defined as:

$$(f \otimes g)(a \otimes x) = (-1)^{|a||g|} \cdot f(a) \otimes g(x).$$

The sign is due to *Koszul's sign convention*: whenever two elements next to each other are swapped (in this case g and a) a minus sign appears if both elements are of odd degree. More formally we can define a swap map

$$\tau : A \otimes B \to B \otimes A : a \otimes b \mapsto (-1)^{|a||b|} b \otimes a.$$

The graded modules together with graded maps of degree 0 form the category **gr-$\Bbbk$Mod** of graded modules. From now on we will simply refer to maps instead of graded maps. Together with the tensor product and the ground ring, (**gr-$\Bbbk$Mod**, $\otimes$, $\Bbbk$) is a symmetric monoidal category (with the symmetry given by τ). This now dictates the definition of a graded algebra.

Definition A.1.5. A *graded algebra* consists of a graded module A together with two maps of degree 0:

$$\mu : A \otimes A \to A \quad \text{and} \quad \eta : k \to A$$

such that μ is associative and η is a unit for μ.

A map between two graded algebra will be called a *graded algebra map* if the map is compatible with the multiplication and unit. Such a map is necessarily of degree 0.

Again these objects and maps form a category, denoted as **gr-$\Bbbk$Alg**. We will denote multiplication by a dot or juxtaposition, instead of explicitly mentioning μ.

Definition A.1.6. A graded algebra A is *commutative* if for all $x, y \in A$

$$xy = (-1)^{|x||y|} yx.$$

A.2 DIFFERENTIAL GRADED ALGEBRA

Definition A.2.1. A *differential graded module* (M, d) is a graded module M together with a map $d : M \to M$ of degree -1, called a *differential*, such that $dd = 0$. A map $f : M \to N$ is a *chain map* if it is compatible with the differential, i.e. $d_N f = f d_M$.

A differential graded module (M, d) with $M_i = 0$ for all $i < 0$ is a *chain complex*. A differential graded module (M, d) with $M_i = 0$ for all $i > 0$ is a *cochain complex*. It will be convenient to define $M^i = M_{-i}$ in the latter case, so that $M = \bigoplus_{n \in \mathbb{N}} M^i$ and d is a map of *upper degree* $+1$.

Definition A.2.2. Let (M, d_M) and (N, d_N) be two differential graded modules, their tensor product $M \otimes N$ is a differential graded module with the differential given by:

$$d_{M \otimes N} = d_M \otimes \mathbf{id}_N + \mathbf{id}_M \otimes d_N.$$

Finally we come to the definition of a differential graded algebra. This will be a graded algebra with a differential. Of course we want this to be compatible with the algebra structure, or stated differently: we want μ and η to be chain maps.

Definition A.2.3. A *differential graded algebra (dga)* is a graded algebra A together with an differential d such that in addition the *Leibniz rule* holds:

$$d(xy) = d(x)y + (-1)^{|x|} x d(y) \quad \text{for all } x, y \in A.$$

In general, a map which satisfies the above Leibniz rule is called a *derivation*. It is not hard to see that the definition of a dga precisely defines the monoidal objects in the category of differential graded modules.

In this thesis we will restrict our attention to dga's M with $M^i = 0$ for all $i < 0$, i.e. non-negatively (cohomologically) graded dga's. We denote the category of these dga's by $\mathbf{DGA}_{\Bbbk}$, the category of commutative dga's (cdga's) will be denoted by $\mathbf{CDGA}_{\Bbbk}$. If no confusion can arise, the ground ring $\Bbbk$ will be suppressed in this notation. These objects are also referred to as *(co)chain algebras*.

Definition A.2.4. An *augmented dga* is a dga A with an map $\epsilon : A \to \Bbbk$. Note that this necessarily means that $\epsilon\eta = \mathbf{id}$.

The above notion is dual to the notion of a pointed objects.

Remark A.2.5. *Note that all the above definitions (i.e. the definitions of graded objects, algebras, differentials, augmentations) are orthogonal, meaning that any combination makes sense. However, keep in mind that we require the structures to be compatible. For example, an algebra with differential should satisfy the Leibniz rule (i.e. the differential should be a map of algebras).*

A.3 HOMOLOGY

Whenever we have a differential module we have $d \circ d = 0$, or put in other words: the image of d is a submodule of the kernel of d. The quotient of the two graded modules will be of interest. Note that the following definition depends on the differential d, however it is often left out from the notation.

Definition A.3.1. Given a differential module (M, d) we define the *homology* of M as:

$$H(M) = \ker(d) / \operatorname{im}(d).$$

If the module has more structure as discussed above, homology will preserve this.

Remark A.3.2. *Let M be a differential module. Then homology preserves the following.*

- *If M is graded, so is $H(M)$, where the grading is given by*

$$H(M)_i = \ker(d\big|_{M_i}) / d(M_{i+1})$$

- *If M has an algebra structure, then so does $H(M)$, given by*

$$[z_1] \cdot [z_2] = [z_1 \cdot z_2]$$

- *If M is a commutative algebra, so is $H(M)$.*
- *If M is augmented, so is $H(M)$.*

Of course the converses need not be true. For example the singular cochain complex associated to a space is a graded differential algebra which is *not* commutative. However, by taking homology one gets a commutative algebra.

Note that taking homology of a differential graded module (or algebra) is functorial. Whenever a map $f : M \to N$ of differential graded modules (or algebras) induces an isomorphism on homology, we say that f is a *quasi isomorphism*.

Definition A.3.3. Let M be a graded module. We say that M is n-reduced if $M_i = 0$ for all $i \leq n$. Similarly we say that a graded augmented algebra A is n-reduced if $A_i = 0$ for all $1 \leq i \leq n$ and $\eta : \Bbbk \xrightarrow{\cong} A_0$.

Let (M, d) be a chain complex (or algebra). We say that M is n-connected if $H(M)$ is n-reduced as graded module (resp. augmented algebra). Similarly for cochain complexes (or algebras).

A.4 CLASSICAL RESULTS

We will give some classical known results of algebraic topology or homological algebra. Proofs of these theorems can be found in many places such as [Rot09, Wei95].

Theorem A.4.1. *(Universal coefficient theorem) Let C be a chain complex and A an abelian group, then there are natural short exact sequences for each n:*

$$0 \to H_n(C) \otimes A \to H_n(C \otimes A) \to Tor(H_{n-1}(C), A) \to 0$$

$$0 \to Ext(H_{n-1}(C), A) \to H^n(\mathbf{Hom}(C, A)) \to \mathbf{Hom}(H_n(C), A) \to 0$$

The first statement generalizes to a theorem where A is a chain complex itself. When choosing to work over a field the torsion will vanish and the exactness will induce an isomorphism. This is (one formulation of) the Künneth theorem.

Theorem A.4.2. *(Künneth) Assume that $\Bbbk$ is a field and let C and D be (co)chain complexes, then there is a natural isomorphism (a linear graded map of degree 0):*

$$H(C) \otimes H(D) \xrightarrow{\cong} H(C \otimes D),$$

where we understand both tensors as graded. If C and D are algebras, this isomorphism is an isomorphism of algebras.

A.5 THE FREE CDGA

Just as in ordinary linear algebra we can form an algebra from any graded module. Furthermore we will see that a differential induces a derivation.

Definition A.5.1. The *tensor algebra* of a graded module M is defined as

$$T(M) = \bigoplus_{n \in \mathbb{N}} M^{\otimes n},$$

where $M^{\otimes 0} = \mathbb{k}$. An element $m = m_1 \otimes \ldots \otimes m_n$ has a *word length* of n and its degree is $|m| = \sum_{i=i}^{n} |m_i|$. The multiplication is given by the tensor product (note that the bilinearity follows immediately).

Note that this construction is functorial and it is free in the following sense.

Lemma A.5.2. *Let M be a graded module and A a graded algebra.*

- *A graded map $f : M \to A$ of degree 0 extends uniquely to an algebra map $\overline{f} : TM \to A$.*
- *A differential $d : M \to M$ extends uniquely to a derivation $d : TM \to TM$.*

Corollary A.5.3. *Let U be the forgetful functor from graded algebras to graded modules, then T and U form an adjoint pair:*

$$T : \textbf{gr-}\mathbb{k}\textbf{Mod} \rightleftarrows \textbf{gr-}\mathbb{k}\textbf{Alg} : U$$

Moreover it extends and restricts to

$$T : \textbf{dg-}\mathbb{k}\textbf{Mod} \rightleftarrows \textbf{dg-}\mathbb{k}\textbf{Alg} : U$$

$$T : \mathbf{Ch}^{\mathbf{n}\geq\mathbf{0}}(\mathbb{k}) \rightleftarrows \mathbf{DGA}\mathbb{k} : U$$

As with the symmetric algebra and exterior algebra of a vector space, we can turn this graded tensor algebra in a commutative graded algebra.

Definition A.5.4. Let A be a graded algebra and define

$$I = \langle ab - (-1)^{|a||b|} ba \mid a, b \in A \rangle$$

Then A/I is a commutative graded algebra.

For a graded module M we define the *free commutative graded algebra* as

$$\Lambda(M) = TM/I$$

Again this extends to differential graded modules (i.e. the ideal is preserved by the derivative) and restricts to cochain complexes.

Lemma A.5.5. *We have the following adjunctions:*

$$\Lambda : \textbf{gr-}\mathbb{k}\textbf{Mod} \rightleftarrows \textbf{gr-}\mathbb{k}\textbf{Alg}^{comm} : U$$

$$\Lambda : \textbf{dg-}\mathbb{k}\textbf{Mod} \rightleftarrows \textbf{dg-}\mathbb{k}\textbf{Alg}^{comm} : U$$

$$\Lambda : \mathbf{Ch}^{\mathbf{n}\geq\mathbf{0}}(\mathbb{k}) \rightleftarrows \mathbf{CDGA}_{\mathbb{k}} : U$$

We can now easily construct cdga's by specifying generators and their differentials. Note that a free algebra has a natural augmentation, defined as $\epsilon(v) = 0$ for every generator v and $\epsilon(1) = 1$.

B

MODEL CATEGORIES

As this thesis considers different categories, each with its own homotopy theory, it is natural to use Quillen's formalism of model categories. Not only gives this the right definition of the associated homotopy category, it also gives existence of lifts and lifts of homotopies.

Definition B.0.6. A *model category* is a category $\mathbf{C}$ together with three subcategories:

- a class of *weak equivalences* $\mathfrak{W}$,
- a class of *fibrations* $\mathfrak{Fib}$ and
- a class of *cofibrations* $\mathfrak{Cof}$,

such that the following five axioms hold:

MC1 All finite limits and colimits exist in $\mathbf{C}$.

MC2 The *2-out-of-3* property: if f, g and fg are maps such that two of them are weak equivalences, then so it the third.

MC3 All three classes of maps are closed under retracts. A class $\mathfrak{K}$ is closed under retracts if, when given a diagram

$$\begin{tikzcd} A' \arrow[r, "i"] \arrow[d, "g"] & A \arrow[r, "r"] \arrow[d, "f"] & A' \arrow[d, "g"] \\ X' \arrow[r, "j"] & X \arrow[r, "s"] & X' \end{tikzcd}$$

with $r \circ i = \mathbf{id}$ and $s \circ j = \mathbf{id}$, then $f \in \mathfrak{K}$ implies $g \in \mathfrak{K}$.

MC4 In any commuting square with $i \in \mathfrak{Cof}$ and $p \in \mathfrak{Fib}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

there exist a lift $h : B \to Y$ if either (a) $i \in \mathfrak{W}$ or (b) $p \in \mathfrak{W}$.

MC5 Any map $f : A \to B$ can be factored in two ways:

a) as $f = pi$, where $i \in \mathfrak{Cof} \cap \mathfrak{W}$ and $p \in \mathfrak{Fib}$ and

b) as $f = pi$, where $i \in \mathfrak{Cof}$ and $p \in \mathfrak{Fib} \cap \mathfrak{W}$.

Notation B.0.7. For brevity

- we write $f : A \twoheadrightarrow B$ when f is a fibration,
- we write $f : A \hookrightarrow B$ when f is a cofibration and
- we write $f : A \xrightarrow{\simeq} B$ when f is a weak equivalence.

Furthermore a map which is a fibration and a weak equivalence is called a *trivial fibration*, similarly we have *trivial cofibration*.

Definition B.0.8. An object A in a model category $\mathbf{C}$ will be called *fibrant* if $A \to \mathbf{1}$ is a fibration and *cofibrant* if $\mathbf{0} \to A$ is a cofibration.

Note that axiom [MC5a] allows us to replace any object X with a weakly equivalent fibrant object X^{fib} and by [MC5b] by a weakly equivalent cofibrant object X^{cof}, as seen in the following diagram:

$$
\begin{tikzcd}
\mathbf{0} \arrow[rr, hook] \arrow[dr, hook] & & X \\
& X^{cof} \arrow[ur, two heads, "\simeq"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, hook, "\simeq"'] & & \mathbf{1} \\
& X^{fib} \arrow[ur, two heads] &
\end{tikzcd}
$$

The fourth axiom actually characterizes the classes of (trivial) fibrations and (trivial) cofibrations. We will abbreviate left lifting property with LLP and right lifting property with RLP. We will not prove these statements, but only expose them because we use them throughout this thesis. One can find proofs in [DS95, May99].

Lemma B.0.9. *Let* $\mathbf{C}$ *be a model category.*

- *The cofibrations in* $\mathbf{C}$ *are the maps with a LLP w.r.t. trivial fibrations.*
- *The fibrations in* $\mathbf{C}$ *are the maps with a RLP w.r.t. trivial cofibrations.*
- *The trivial cofibrations in* $\mathbf{C}$ *are the maps with a LLP w.r.t. fibrations.*
- *The trivial fibrations in* $\mathbf{C}$ *are the maps with a RLP w.r.t. cofibrations.*

This means that once we choose the weak equivalences and the fibrations for a category $\mathbf{C}$, the cofibrations are determined, and vice versa. The classes of fibrations behave nice with respect to pullbacks and dually cofibrations behave nice with pushouts:

Lemma B.0.10. *Let* **C** *be a model category. Consider the following two diagrams where P is the pushout and pullback respectively.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "i", "(\simeq)"'] & C \arrow[d, "j"] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & P \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p", "(\simeq)"'] \\ Z \arrow[r] & Y \end{tikzcd}$$

- *If i is a (trivial) cofibration, so is j.*
- *If p is a (trivial) fibration, so is q.*

Lemma B.0.11. *Let* **C** *be a model category. Let $f : A \hookrightarrow B$ and $g : A' \hookrightarrow B'$ be two (trivial) cofibrations, then the induced map of the coproducts $f + g : A + A' \to B + B'$ is also a (trivial) cofibration. Dually: the product of two (trivial) fibrations is a (trivial) fibration.*

Of course the most important model category is the one of topological spaces. We will be interested in the standard model structure on topological spaces, which has weak homotopy equivalences as weak equivalences. Equally important is the model category of simplicial sets.

Example B.0.12. The category **Top** of topological spaces admits a model structure as follows.

- Weak equivalences: maps inducing isomorphisms on all homotopy groups.
- Fibrations: Serre fibrations, i.e. maps with the right lifting property with respect to the inclusions $D^n \hookrightarrow D^n \times I$.
- Cofibrations: the smallest class of maps containing $S^{n-1} \hookrightarrow D^n$ which is closed under transfinite compositions, pushouts, coproducts and retracts.

Example B.0.13. The category **sSet** of simplicial sets has the following model structure.

- Weak equivalences: maps inducing isomorphisms on all homotopy groups.
- Fibrations: Kan fibrations, i.e. maps with the right lifting property with respect to the inclusions $\Lambda^k_n \hookrightarrow \Delta[n]$.
- Cofibrations: all monomorphisms.

Both of these examples are often proven to be model categories by using *Quillen's small object arguments*. This technique can be found in [GS06, DS95, MP11].

In this thesis we often restrict to 1-connected spaces. The full subcategory $\mathbf{Top}_1$ of 1-connected spaces satisfies MC2-MC5: the 2-out-of-3 property, retract property and lifting properties hold as we take the *full* subcategory, factorizations exist as the middle space is 1-connected as well. Both products and coporducts exist. However $\mathbf{Top}_1$ does not have all limits and colimits.

Remark B.0.14. *Let $r > 0$ and $\mathbf{Top}_r$ be the full subcategory of r-connected spaces. The diagrams*

$$\begin{tikzcd} * \arrow[r, shift left, "*\mapsto 0"] \arrow[r, shift right, "*\mapsto 1"'] & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, shift left, "x\mapsto(x,sin(\pi x))"] \arrow[r, shift right, "x\mapsto(x,-sin(\pi x))"'] & \mathbb{R}^2 \end{tikzcd}$$

have no coequalizer and respectively no equalizer in $\mathbf{Top}_r$.

B.1 HOMOTOPIES

So far we have only seen equivalences between objects of the category. We can, however, also define homotopy relations between maps (as we are used to in **Top**). There are two such construction, which will coincide on nice objects. We will only state the definitions and important results. One can find proofs of these results in [DS95]. Throughout this section we silently work with a fixed model category **C**.

Definition B.1.1. A *cylinder object* for an object A is an object Cyl_A together with maps:

$$A \coprod A \xrightarrow{i} Cyl_A \xrightarrow{\simeq}^{p} A,$$

which factors the folding map $\mathbf{id}_A + \mathbf{id}_A : A \coprod A \to A$ (note that we use MC1 here). The cylinder object is called

- *good* if i is a cofibration and
- *very good* if in addition p is a fibration.

Notation B.1.2. The map i consists of two factors, which we will denote i_0 and i_1.

Note that we do not require cylinder objects to be functorial. There can also be more than one cylinder object for A. Cylinder objects can now be used to define left homotopies.

Definition B.1.3. Two maps $f, g : A \to X$ are *left homotopic* if there exists a cylinder object Cyl_A and a map $H : Cyl_A \to X$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

We will call H a *left homotopy* for f to g and write $f \sim^l r$. Moreover, the homotopy is called good (resp. very good) is the cylinder object is good (resp. very good).

Note that the relation need not be transitive: consider $f \sim^l g$ and $g \sim^l h$, then these homotopies may be defined on different cylinder objects and in general we cannot relate the cylinder objects. However for nice domains $\sim^l$ will be an equivalence relation.

Lemma B.1.4. *If A is cofibrant, then $\sim^l$ is an equivalence relation on $\mathbf{Hom_C}(A, X)$.*

Definition B.1.5. We will denote the set of *left homotopy classes* as

$$\pi^l(A, X) = \mathbf{Hom_C}(A, X) / \sim^{l'},$$

where $\sim^{l'}$ is the equivalence relation generated by $\sim^l$.

Lemma B.1.6. *We have the following properties*

- *If A is cofibrant and $p : X \to Y$ a trivial fibration, then*

$$p_* : \pi^l(A, X) \xrightarrow{\cong} \pi^l(A, Y).$$

- *If X is fibrant, $f \sim^l g : B \to X$ and we have a map $h : A \to B$, then*

$$fh \sim^l gh.$$

Of course there is a completely dual definition of right homotopy, in terms of path objects. All of the above also applies (but in a dual way).

Definition B.1.7. A *path object* for an object X is an object $Path_X$ together with maps:

$$X \xrightarrow{\simeq}{}^{i} Path_X \xrightarrow{p} X \times X,$$

which factors the diagonal map $(\mathbf{id}_X, \mathbf{id}_X) : X \to X \times X$. The path object is called

- *good* if p is a fibration and
- *very good* if in addition i is a cofibration.

Notation B.1.8. The map p consists of two factors, which we will denote p_0 and p_1.

Definition B.1.9. Two maps $f, g : A \to X$ are *right homotopic* if there exists a path object $Path_X$ and a map $H : A \to Path_X$ such that $p_0 \circ H = f$ and $p_1 \circ H = g$.

We will call H a *right homotopy* for f to g and write $f \sim^r r$. Moreover, the homotopy is called good (resp. very good) is the path object is good (resp. very good).

Lemma B.1.10. *If X is fibrant, then $\sim^r$ is an equivalence relation on* $\mathbf{Hom_C}(A, X)$.

Definition B.1.11. We will denote the set of *left homotopy classes* as

$$\pi^r(A, X) = \mathbf{Hom_C}(A, X)/\sim^{r\prime},$$

where $\sim^{r\prime}$ is the equivalence relation generated by $\sim^r$.

Lemma B.1.12. *We have the following properties*

- *If X is fibrant and $i : A \to B$ a trivial cofibration, then*

$$i^* : \pi^r(B, X) \xrightarrow{\cong} \pi^r(A, X).$$

- *If A is cofibrant, $f \sim^r g : A \to X$ and we have a map $h : X \to Y$, then*

$$hf \sim^r hg.$$

The two notions (left resp. right homotopy) agree on nice objects. Hence in this case we can speak of homotopic maps.

Lemma B.1.13. *Let $f, g : A \to X$ be two maps and A cofibrant and X fibrant, then*

$$f \sim^l g \iff f \sim^r g.$$

Definition B.1.14. In the above case we say that f and g are *homotopic*, this is denoted by $f \sim g$. Furthermore we can define the set of homotopy classes as:

$$[A, X] = \mathbf{Hom_C}(A, X)/\sim.$$

A map $f : A \to X$ between cofibrant-fibrant objects is said to have a *homotopy inverse* if there exists $g : X \to A$ such that $fg \sim \mathbf{id}$ and $gf \sim \mathbf{id}$. We will also call f a *strong homotopy equivalence*.

Lemma B.1.15. *Let $f : A \to B$ be a map between cofibrant-fibrant objects, then:*

$$f \textit{ is a weak equivalence} \iff f \textit{ is a strong equivalence}.$$

B.2 THE HOMOTOPY CATEGORY

Given a model category, we wish to construct a category in which the weak equivalences become actual isomorphisms. From an abstract perspective, this would be a *localization* of categories. To be precise, if we have a category $\mathbf{C}$ with weak equivalences $\mathfrak{W}$, we want a functor $\gamma : \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ such that

- for every $f \in \mathfrak{W}$, the map $\gamma(f)$ is an isomorphism and
- $\mathbf{Ho}(\mathbf{C})$ is universal with this property. This means that for every ψ sending weak equivalences to isomorphisms, we get:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\gamma"] \arrow[dr, "\psi"'] & \mathbf{Ho}(\mathbf{C}) \arrow[d, dashed, "\overline{\psi}"] \\ & \mathbf{D} \end{tikzcd}$$

For arbitrary categories and classes of weak equivalences, such a localization need not exist. But when we have a model category, we can always construct $\mathbf{Ho}(\mathbf{C})$.

Definition B.2.1. The *homotopy category* $\mathbf{Ho}(\mathbf{C})$ of a model category $\mathbf{C}$ is defined with

- the objects being the objects in $\mathbf{C}$
- the maps between X and Y are

$$\mathbf{Hom}_{\mathbf{Ho}(\mathbf{C})}(X, Y) = [X^{cof,fib}, Y^{cof,fib}].$$

In [DS95] it is proven that this indeed defines a localization of $\mathbf{C}$ with respect to $\mathfrak{W}$. It is good to note that $\mathbf{Ho}(\mathbf{Top})$ does not depend on the class of cofibrations or fibrations.

Note that whenever we have a full subcategory $\mathbf{C}' \subset \mathbf{C}$, where $\mathbf{C}$ is a model category, there is a subcategory of the homotopy category: $\mathbf{Ho}(\mathbf{C}') \subset \mathbf{Ho}(\mathbf{C})$. There is no need for a model structure on the subcategory.

Example B.2.2. The category $\mathbf{Ho}(\mathbf{Top})$ has as objects just topological spaces and homotopy classes between cofibrant replacements (note that every objects is already fibrant). Moreover, if we restrict to the full subcategory of CW complexes, maps are precisely homotopy classes between objects.

Homotopical invariants are often defined as functors on $\mathbf{Top}$. For example we have the n-th homotopy group functor

$\pi_n : \mathbf{Top} \to \mathbf{Grp}$ and the n-th homology group functor $H_n : \mathbf{Top} \to \mathbf{Ab}$. But since they are homotopy invariant, they can be expressed as functors on $\mathbf{Ho}(\mathbf{Top})$:

$$\pi_n : \mathbf{Ho}(\mathbf{Top}) \to \mathbf{Grp} \qquad H_n : \mathbf{Ho}(\mathbf{Top}) \to \mathbf{Ab}.$$

B.3 QUILLEN PAIRS

In order to relate model categories and their associated homotopy categories we need a notion of maps between them. We want the maps such that they induce maps on the homotopy categories.

We first make an observation. Notice that whenever we have an adjunction $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$, finding a lift in the following diagram on the left is equivalent to finding one in the diagram on the right.

$$\begin{array}{ccc} FA & \longrightarrow & X \\ \downarrow & & \downarrow \\ FB & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & GX \\ \downarrow & & \downarrow \\ B & \longrightarrow & GY \end{array}$$

So it should not come as a surprise that adjunctions play an important role in model categories. The useful notion of maps between model categories is the following.

Definition B.3.1. An adjunction $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ between model categories is a *Quillen pair* if F preserves cofibrations and G preserves fibrations.

In this case F is the *left Quillen functor* and G is the *right Quillen functor*.

Notice that by the lifting properties (F, G) is a Quillen pair if and only if F preserves cofibrations and trivial cofibrations (or dually G preserves fibrations and trivial fibrations). The Quillen pairs are important as they induce functors on the homotopy categories.

Theorem B.3.2. *If (F, G) is a Quillen pair, then there an induced adjunction*

$$LF : \mathbf{Ho}(catC) \rightleftarrows \mathbf{Ho}(\mathbf{D}) : RG,$$

where $LF(X) = F(X^{cof})$ and $RG(Y) = G(Y^{fib})$.

Such an adjunction between homotopy categories is an equivalence if the unit and counit are isomorphisms in $\mathbf{Ho}(\mathbf{C})$. This means that the following two maps should be weak equivalences in $\mathbf{C}$ for all cofibrant X and all fibrant Y

$$\eta : X \to G(F(X)^{fib})$$
$$\epsilon : F(G(Y)^{cof}) \to Y.$$

In this case, such a pair of functors is called a *Quillen equivalence.*

Example B.3.3. The geometric realization and singular functor form a Quillen equivalence

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S(-).$$

B.4 HOMOTOPY PUSHOUTS AND PULLBACKS

In category theory we know that colimits (and limits) are unique up to isomorphism, and that isomorphic diagrams will have isomorphic colimits (and limits). We would like a similar theory for weak equivalences. Unfortunately the ordinary colimit (or limit) is not homotopically nice. For example consider the following two diagrams, with the obvious maps.

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \\ D^2 & \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

The diagrams are pointwise weakly equivalent. But the induced map $S^n \to *$ on the pushout is clearly not. In this section we will briefly indicate what homotopy pushouts are (and dually we get homotopy pullbacks).

One direct way to obtain a homotopy pushout is by the use of *Reedy categories* [Hov07]. In this case the diagram category is endowed with a model structure, which gives a notion of cofibrant diagram. In such diagrams the ordinary pushout is the homotopy pushout.

Lemma B.4.1. *Consider the following pushout diagram. The if all objects are cofibrant and the map f is a cofibration, then the homotopy pushout is given by the ordinary pushout.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & C \arrow[d] \\ B \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

There are other ways to obtain homotopy pushouts. A very general way is given by the *bar construction* [Rie14].

The important property of homotopy pushout we use in this thesis is the uniqueness (up to homotopy). In particular we need the following fact.

Lemma B.4.2. *(The cube lemma) Consider the following commuting diagram, where P and Q are the homotopy pushouts of the back and front face respectively.*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & A' \arrow[dr] \arrow[dd] & \\
& B \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
A'' \arrow[rr, crossing over] \arrow[dr] & & P \arrow[dr] & \\
& B'' \arrow[rr] & & Q
\end{tikzcd}$$

If the three maps $A^* \to B^*$ are weak equivalences, then so is the map $P \to Q$.

If we combine this lemma with Lemma B.4.1 we obtain precisely Lemma 5.2.6 in [Hov07]. We get similar theorems for the dual case of homotopy pullbacks.

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